Branching Brownian motion with absorption and the all-time minimum of branching Brownian motion with drift

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Abstract

We study a dyadic branching Brownian motion on the real line with absorption at 0, drift $\mu \in \mathbb{R}$ and started from a single particle at position $x > 0$. With $K(\infty)$ the (possibly infinite) total number of individuals absorbed at 0 over all time, we consider the functions $\omega_s(x) := \mathbb{E}^x[s^{K(\infty)}]$ for $s \geq 0$. In the regime where μ is large enough so that $K(\infty) < \infty$ almost surely and that the process has a positive probability of survival, we show that $\omega_s < \infty$ if and only if $s \in [0, s_0]$ for some $s_0 > 1$ and we study the properties of these functions. Furthermore, $\omega(x) := \omega_0(x) = \mathbb{P}^x(K(\infty) = 0)$ is the cumulative distribution function of the all time minimum of the branching Brownian motion with drift started at 0 without absorption.

We give descriptions of the family $\omega_s, s \in [0, s_0]$ through the single pair of functions $\omega_0(x)$ and $\omega_{s_0}(x)$, as extremal solutions of the Kolmogorov-Petrovskii-Piskunov (KPP) traveling wave equation on the half-line, through a martingale representation, and as a single explicit series expansion. We also obtain a precise result concerning the tail behavior of $K(\infty)$. In addition, in the regime where $K(\infty) > 0$ almost surely, we show that $u(x,t) := \mathbb{P}^x(K(t) = 0)$ suitably centered converges to the KPP critical travelling wave on the whole real line.

1 Introduction

Consider a branching Brownian motion in which particles move according to a Brownian motion with drift $\mu \in \mathbb{R}$ and split into two particles at rate β independently one from another. Call $\mathcal{N}_{\text{all}}(t)$ the population of all particles at time t and call $X_u(t)$ the position of a given particle $u \in \mathcal{N}_{\text{all}}(t)$. When we start with a single particle at position x we write $\mathbb{P}^x$ for the law of this process.

In a seminal paper, [21], Kesten considered the branching Brownian motion with absorption, that is, the model just described with the additional property that particles entering the negative half-line $(-\infty, 0]$ are immediately absorbed and removed. We write $\mathcal{N}_{\text{live}}(t)$ for the set of particles alive (not absorbed) in the branching Brownian motion with absorption and $K(t)$ the number of particles that have been absorbed up to time t. The system with absorption is said to become extinct if $\exists t \geq 0 : \mathcal{N}_{\text{live}}(t) = \emptyset$ and to survive otherwise. We let $K(\infty) := \lim_{t\to\infty} K(t) \in \mathbb{N} \cup \{\infty\}$.

Depending on the value of μ one has the following behaviours (see Figure 1)

Regime A: if $\mu \leq -\sqrt{2\beta}$, the drift towards origin is so large that the system goes extinct almost surely. $K(\infty)$ is finite and non-zero.

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Regime B: if $-\sqrt{2\beta} < \mu < \sqrt{2\beta}$ there is a non-zero probability of survival. On survival, there will always be particles near 0 and $K(\infty) = \infty$ almost surely.

Regime C: if $\mu \geq \sqrt{2\beta}$ there is still a non-zero probability of survival, but the system is drifting so fast away from 0 that, on survival, $\min_{u \in \mathcal{N}_{\text{all}}(t)} X_u(t)$ drifts to $+\infty$ almost surely as $t \to \infty$; $K(\infty)$ is thus almost surely finite. Furthermore, there is a non-zero probability that $K(\infty) = 0$.

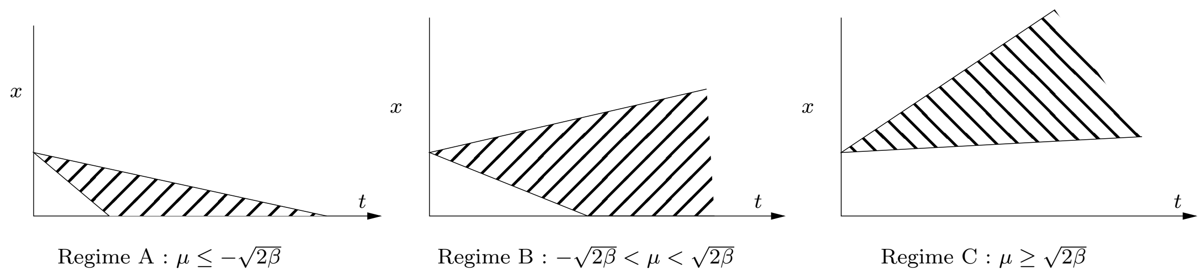

Figure 1: The three regimes: The hashed region represents the cone in which one expects to find particles.

The behaviour of $K(\infty)$ in regime A ($\mu \leq -\sqrt{2\beta}$) has been the subject of very active research recently, including a conjecture by Aldous which was recently settled by P. Maillard [24] (we discuss Maillard's results below), improving earlier results of L. Addario-Berry and N. Broutin [1] and E. Aïdékon [2]. Surprisingly, relatively little was known concerning the regimes B and C. Our main results in the present work concern the study of $K(\infty)$ in regime C and of certain related KPP-type equations.

1.1 The tail behaviour of $K(\infty)$

In [24], Pascal Maillard has shown that, in regime A ($\mu \leq -\sqrt{2\beta}$), the variable $K(\infty)$ has a very fat tail. More precisely, it is shown there that, as $z \to \infty$, there exists two constants c, c' which depend on x such that

$$\mathbb{P}^x[K(\infty) > z] \sim \begin{cases} c/(z \log(z)^2) & \text{for } \mu = -\sqrt{2\beta}, \\ c' z^{-a(\mu)} & \text{for } \mu < -\sqrt{2\beta} \text{ where } a(\mu) = \frac{\mu + \sqrt{\mu^2 - 2\beta}}{\mu - \sqrt{\mu^2 - 2\beta}}. \end{cases}$$

In regime B ($-\sqrt{2\beta} < \mu < \sqrt{2\beta}$) it is clear that $K(\infty) = \infty$ on survival so one would essentially need to condition on extinction to study the tail behaviour of $K(\infty)$. This is outside the scope of the present work.

In regime C ($\mu \geq \sqrt{2\beta}$), however, $K(\infty)$ is almost surely finite. We introduce for $s \geq 0$ and $x \geq 0$,

$$\omega_s(x) := \mathbb{E}^x\left[s^{K(\infty)}\right], \quad \omega_s(0) = s. \tag{1}$$

When $s \in [0, 1]$ this quantity is the generating function of $K(\infty)$. We show that $\omega_s(x)$ is finite for some values of s larger than 1:

Theorem 1. *In regime C ($\mu \geq \sqrt{2\beta}$), there exists a finite $s_0 > 1$ depending only on $\mu/\sqrt{\beta}$ such that*

1. *For $s \leq s_0$, $\omega_s(x)$ is finite for all $x \geq 0$,*
2. *For $s > s_0$, $\omega_s(x)$ is infinite for all $x > 0$.*

The functions $x \mapsto \omega_s(x)$ are increasing for any $s \in [0, 1)$ and decreasing for any $s \in (1, s_0]$, converging to 1 when $x \to \infty$, and one has $\omega'_{s_0}(0) = 0$.

Furthermore, one has for n large

$$\mathbb{P}^x[K(\infty) = n] \sim \frac{-\omega'_{s_0}(x)}{2s_0^n n^{\frac{3}{2}}\sqrt{\pi\beta(s_0 - 1)}}. \tag{2}$$

A familiar argument using the branching structure and a simple coupling allows us to relate the $\omega_s(x)$ with one another, as in the following result (which is already present in [26, 24]):

Theorem 2. *In regime C ($\mu \geq \sqrt{2\beta}$),*

1. *For each $s \in [0, 1)$,* $\quad \omega_s(x) = \omega_0(x + \omega_0^{-1}(s))$,
2. *For each $s \in (1, s_0]$,* $\quad \omega_s(x) = \omega_{s_0}(x + \omega_{s_0}^{-1}(s))$.

Although we do not have an explicit expression for s_0 as a function of $\mu/\sqrt{\beta}$, we can evaluate it numerically with a good precision as shown in Figure 2. In the critical case $\mu = \sqrt{2\beta}$, we obtain $s_0 = 1.3486\ldots$.

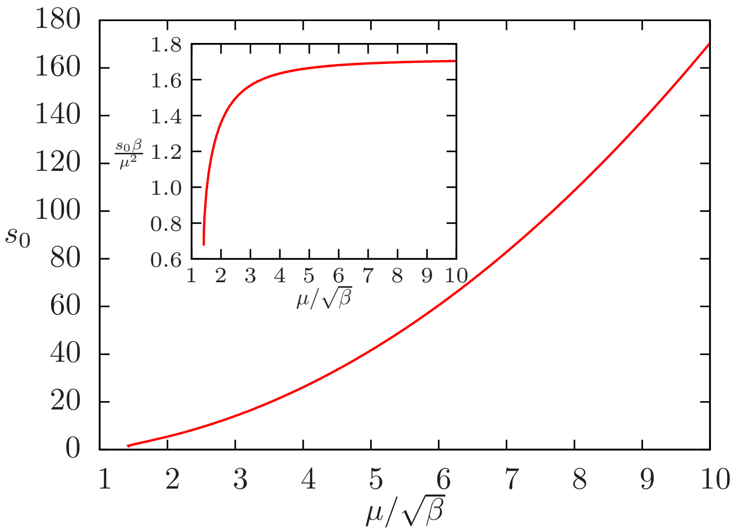

Figure 2: Numerical determination of s_0 and $s_0\beta/\mu^2$ as functions of $\mu/\sqrt{\beta}$.

We also prove the following property:

Proposition 3. *s_0 is an increasing function of $\mu/\sqrt{\beta}$ and furthermore $s_0 \sim c\mu^2/\beta$ for some constant c as $\mu/\sqrt{\beta} \to \infty$.*

1.2 Distribution of the all-time minimum in a branching Brownian motion

The probability that $K(\infty) = 0$ for a system started from x, is also the probability that the all-time minimum of a full branching Brownian motion with drift μ started from zero does not go below $-x$:

$$\omega(x) := \omega_0(x) = \mathbb{P}^x[K(\infty) = 0] = \mathbb{P}^0[\min_{t\geq 0} \min_{u\in\mathcal{N}_{\text{all}}(t)} X_u(t) > -x]. \tag{3}$$

This quantity, of course, is non-trivial only in regime C ($\mu \geq \sqrt{2\beta}$). Then, since

$$\lim_{t\to\infty} \min_{u\in\mathcal{N}_{\text{all}}(t)} X_u(t) = +\infty$$

almost surely, we see that there is an almost surely finite all-time minimum for the branching Brownian motion and we conclude that $\lim_{x\to\infty}\omega(x)=1$.

It is not hard to see by standard arguments that ω must satisfy a KPP-type differential equation with boundary conditions:

$$\begin{cases} 0=\frac{1}{2}\omega''+\mu\omega'+\beta(\omega^2-\omega), & x\geq 0,\\ \omega(0)=0, \quad \omega(\infty)=1. \end{cases} \tag{D_0}$$

In fact, $\omega_s(x)$ introduced in (1), if finite, is solution to the same equation with the boundary condition $\omega(0)=0$ replaced by $\omega_s(0)=s$:

$$\begin{cases} 0=\frac{1}{2}\omega_s''+\mu\omega_s'+\beta(\omega_s^2-\omega_s), & x\geq 0,\\ \omega_s(0)=s, \quad \omega_s(\infty)=1. \end{cases} \tag{D_s}$$

This is an example of the deep connection between branching Brownian motion and the KPP equation which goes back to McKean [25] who noticed that one can represent solutions of the KPP equation as expectations of functionals of branching Brownian motions.

Until now this is very classical, however there is one unexpected difficulty here: both (D_0) and (D_s) admit infinitely many solutions and are not sufficient to characterize $\omega(x)$. Figure 3 gives several numerical solutions to (D_0).

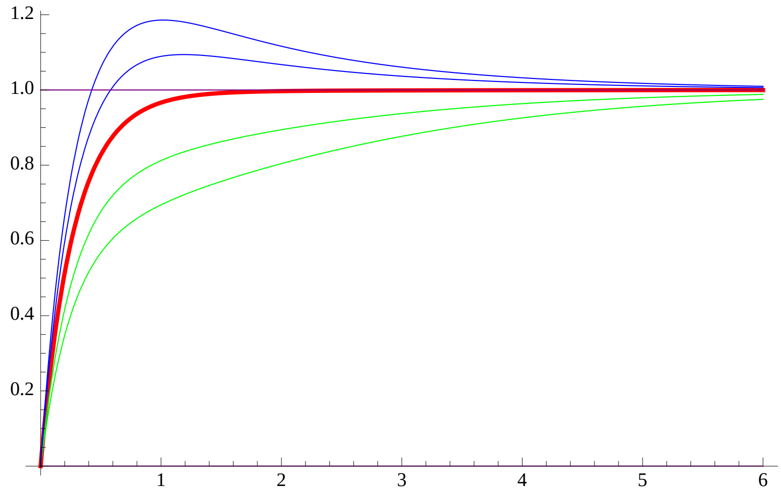

Figure 3: Solutions to (D_0) for $\beta=1$ and $\mu=2$. The red (bold) curve is ω.

In this work, we present three largely independent ways to characterize $\omega(x)$ which are laid out in the three following subsections. The first approach relies on partial differential equations, the second gives $\omega(x)$ as the expectation of a certain martingale and the third one gives $\omega(x)$ as a power series.

One salient property of ω is that it converges to 1 rather quickly:

Proposition 4. *There exists $B>0$ such that*

$$1-\omega(x)\sim Be^{-\left(\mu+\sqrt{\mu^2-2\beta}\right)x} \qquad \text{for large } x. \tag{4}$$

Furthermore, $\omega(x)$ is the only solution of (D_0) which remains in $[0,1)$ and converges that fast to 1.

Similarly, for any $s\in[0,s_0)$, there exists $B_s\in\mathbb{R}$ such that

$$1-\omega_s(x)\sim B_se^{-\left(\mu+\sqrt{\mu^2-2\beta}\right)x} \qquad \text{for large } x \tag{5}$$

where $B_1=0$, $B_s>0$ when $s<1$ and $B_s<0$ when $s>1$.

The following Corollary is a direct consequence of (4) and (5):

Corollary 5. *We have*

$$\lim_{x\to\infty} \mathbb{E}^x[s^{K(\infty)}|K(\infty)>0] = 1 - \frac{B_s}{B}. \tag{6}$$

Hence, as $x \to \infty$, conditionally on $K(\infty) > 0$, the variable $K(\infty)$ converges in distribution and the limit has probability generating function $s \mapsto 1 - \frac{B_s}{B}$.

Remarks:

- The fast decay in (4) and (5) was actually an unexpected result for reasons explained in Section 1.2.4. To simplify notation, we call r the exponential decay rate of $1 - \omega(x)$ as given in (4):
$$r := \mu + \sqrt{\mu^2 - 2\beta}. \tag{7}$$
It is the largest solution of $\frac{1}{2}r^2 - \mu r + \beta = 0$.
- The convergence (6) can actually be thought of as a Yaglom type result. Indeed, imagine that we start with a single particle at the origin and that we kill particles when they first hit $-x$. We call K_x the total number of particles absorbed at $-x$. Clearly by translation invariance this is the same as starting from x and killing at zero. Thus K_x has same distribution as $K(\infty)$ under $\mathbb{P}^x$. Indeed, with this method we can think of $(K_x, x \geq 0)$ as a continuous-time Galton-Watson process indexed by x. This approach is by now classical; it was pioneered by Neveu [26] and recently put to fruitful use by Maillard [24]. Somewhat surprisingly, even when $\mu = \sqrt{2\beta}$ this Galton-Watson process is sub-critical (with $\mathbb{E}[K_x] = e^{-rx}$, see Lemma 7). Thus r is the rate at which the survival probabilty decreases for this Galton-Watson process and (6) is the corresponding Yaglom limit.

1.2.1 $\omega(x)$ from partial differential equations

A first way to characterize $\omega(x)$ is to track the probability that no particle has been absorbed up to time t. Define

$$u(t,x) := \mathbb{P}^x[K(t) = 0]. \tag{8}$$

The function $u : \mathbb{R}_+^2 \mapsto [0,1]$ is increasing in x and decreasing in t and, clearly, for each x, $u(t,x) \to \omega(x)$ as $t \to \infty$. Furthermore, u satisfies the KPP equation with boundary conditions

$$\begin{cases} \partial_t u = \frac{1}{2}\partial_{xx}u + \mu\partial_x u + \beta(u^2 - u), \\ u(t,0) = 0 \ (\forall t \geq 0), \quad u(0,x) = 1 \ (\forall x > 0), \end{cases} \tag{9}$$

which, by Cauchy's Theorem has only one solution.

Therefore, to obtain $\omega(x)$, one can in principle solve (9) and take the large time limit. This route leads to the following Theorem:

Theorem 6. *In regime C ($\mu \geq \sqrt{2\beta}$), the function $\omega(x)$ defined by* (3) *is the maximal solution of* (D_0) *such that $\omega(x) < 1$ for all $x \geq 0$.*

More generally, when $s < 1$, $\omega_s(x)$ is the maximal solution of (D_s) that stays below 1. When $s \in (1, s_0]$, $\omega_s(x)$ is the minimal solution of (D_s) that stays above 1.

1.2.2 Martingale representation for ω

There is an explicit probabilistic representation of the maximum standing wave ω in regime C ($\mu \geq \sqrt{2\beta}$). Recall that $\mathcal{N}_{\mathrm{all}}(t)$ is the population of all the particles in the branching Brownian motion with no absorption and $\mathcal{N}_{\mathrm{live}}(t)$ is the population of particles alive at time t when we kill at 0. We now define on the same probability space a third process based on the branching

Brownian motion in which particles that hit 0 are stopped but not removed from the system (they neither move nor branch). We denote by $\mathcal{N}_{\text{live+abs.}}(t)$ the set of particles in the system at time t in this model. With a slight abuse of notations we continue to write $X_u(t)$ for the positions of particles when $u \in \mathcal{N}_{\text{live+abs.}}(t)$.

Let us define the following two processes:

$$Z_{\text{live+abs.}}(t) := \sum_{u \in \mathcal{N}_{\text{live+abs.}}(t)} e^{-rX_u(t)}, \qquad Z_{\text{all}}(t) := \sum_{u \in \mathcal{N}_{\text{all}}(t)} e^{-rX_u(t)}, \tag{10}$$

where r is the asymptotic decay of $\omega(x)$ as given in (7). Rewriting $X_u(t) = Y_u(t) + \mu t$ it is clear that $\{Y_u(t), u \in \mathcal{N}_{\text{all}}(t)\}$ is simply a standard branching Brownian motion with no drift. Therefore Z_{all} is nothing other than the usual exponential martingale with parameter r associated with the branching Brownian motion Y. Now, the process $Z_{\text{live+abs.}}$ can be thought of as the martingale Z_{all} stopped on the lines $t \wedge T_0$ (that is, where individual particles are stopped at the earlier of either time t or the time when they first hit 0) and therefore it is also a martingale (one can check directly, or more generally see Chauvin [10]).

Lemma 7. *In regime C ($\mu \geq \sqrt{2\beta}$) the martingale $(Z_{\text{live+abs.}}(t), t \geq 0)$ converges almost surely and in L^1 to $K(\infty)$ and therefore $\mathbb{E}^x[K(\infty)] = Z_{\text{live+abs.}}(0) = e^{-rx}$.*

Following the probability tilting method pioneered by [23] and [11] we introduce a new probability measure $\mathbb{Q}^x$

$$\frac{\mathrm{d}\mathbb{Q}^x}{\mathrm{d}\mathbb{P}^x} = e^{rx} K(\infty).$$

Note that since $Z_{\text{live+abs.}}$ is a closed martingale we have that $\mathbb{E}^x[K(\infty)|\mathcal{F}_t] = Z_{\text{live+abs.}}(t)$. Thus

$$\left.\frac{\mathrm{d}\mathbb{Q}^x}{\mathrm{d}\mathbb{P}^x}\right|_{\mathcal{F}_t} = \frac{Z_{\text{live+abs.}}(t)}{Z_{\text{live+abs.}}(0)}.$$

Under this tilted probability measure, the law of the process is the same as the original $\mathbb{P}^x$ law except for the movement and branching rate of a distinguished particle (the *spine* particle ξ). The spine moves according to a Brownian motion with drift $-\sqrt{\mu^2 - 2\beta}$, branches at an accelerated rate of 2β and stops (i.e. sticks and stops reproducing) upon hitting 0.

Theorem 8. *In regime C ($\mu \geq \sqrt{2\beta}$),*

$$1 - \omega(x) = \mathbb{Q}^x\left(\frac{1}{K(\infty)}\right) e^{-rx}.$$

Furthermore, $\mathbb{Q}^x\left(\frac{1}{K(\infty)}\right)$ converges to a finite constant $B > 0$ when $x \to \infty$ and thus, as $x \to \infty$,

$$1 - \omega(x) \sim B e^{-rx}.$$

More generally, for any $s \in [0, s_0]$, one has

$$1 - \omega_s(x) = \mathbb{Q}^x\left(\frac{1 - s^{K(\infty)}}{K(\infty)}\right) e^{-rx}$$

and the expectation $\mathbb{Q}^x(\cdot)$ converges to a finite positive constant as $x \to \infty$.

We will see in the proof that we can give an explicit representation of the constant B which appears in Proposition 4 as the expectation of $K(\infty)^{-1}$ under the measure $\mathbb{Q}^\infty$ (similar to $\mathbb{Q}^x$ but with the spine particle "started at infinity").

1.2.3 $\omega(x)$ as a series expansion

The function $\omega(x)$ can be understood in terms of series expansion. Let $\{a_n\}_{n\geq 1}$ be the sequence defined by

$$a_1 = 1, \qquad a_n = \frac{\beta}{\frac{1}{2}n^2r^2 - n\mu r + \beta}\sum_{j=1}^{n-1} a_j a_{n-j} = \frac{1}{(n-1)\left(\frac{r^2}{2\beta}n - 1\right)}\sum_{j=1}^{n-1} a_j a_{n-j}, \quad n \geq 2, \tag{11}$$

(recall that r was defined in (7), that $\frac{1}{2}r^2 - \mu r + \beta = 0$ and that $r \geq \mu \geq \sqrt{2\beta}$) and let Φ be the function defined by the series

$$\Phi(z) = \sum_{n\geq 1} a_n z^n. \tag{12}$$

We have

Proposition 9. *The radius of convergence $\mathcal{R}$ of Φ is non-zero and there exists $B \in (0, \mathcal{R})$ such that*

$$\omega(x) = 1 - \Phi(Be^{-rx}).$$

More generally, for any $0 \leq s \leq s_0$, there exists a number B_s such that

$$\omega_s(x) = 1 - \Phi(B_s e^{-rx}) \qquad \textit{for all } x \geq 0 \textit{ such that } |B_s|e^{-rx} < \mathcal{R}. \tag{13}$$

$s \mapsto B_s$ is decreasing, positive for $s < 1$, zero for $s = 1$, and negative for $s > 1$. In particular, for $s \leq 1$, the condition $|B_s|e^{-rx} < \mathcal{R}$ is automatically fulfilled.

The proof is contained in Section 2.2.4. Numerically, it seems that $\mathcal{R}$ is large enough that $|B_s|e^{-rx} < \mathcal{R}$ for all $s \in [0, s_0]$ and all $x \geq 0$, but we haven't proved that point. The representation (13) makes it very easy to compute numerically ω_s by first computing $\Phi(z)$, see Figure 4; the value s_0 is then obtained as 1 minus the first minimum of Φ for negative arguments. This follows easily from the facts that $w'_{s_0}(0) = 0, w''_{s_0}(0) < 0$ and that $w'_s(0) < 0$ for all $s \in (1, s_0)$.

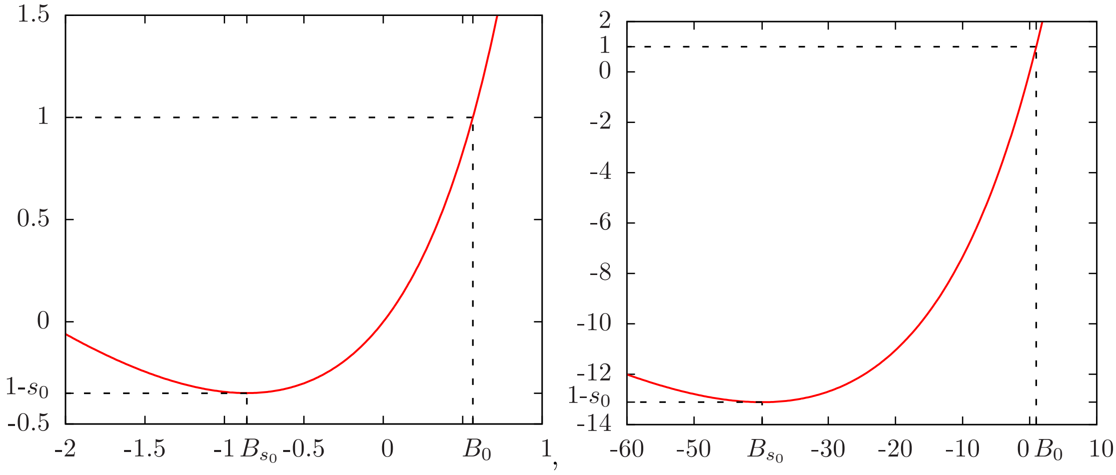

Figure 4: The function $z \mapsto \Phi(z)$ for $\beta = 1$ and $\mu = \sqrt{2}$ (left) or $\mu = 3$ (right). Numerically, one has $s_0 \approx 1.35$, $B_{s_0} \approx -0.859$, $B_0 \approx 0.564$, $\mathcal{R} \approx 3.14$ for $\mu = \sqrt{2}$ and $s_0 \approx 14.11$, $B_{s_0} \approx -39.86$, $B_0 \approx 0.969$, $\mathcal{R} \approx 72.8$ for $\mu = 3$.

1.2.4 Discussion and background

As we have already mentioned the Dirichlet problem (D_s) is a version of the KPP travelling wave equation on the half-line with boundary condition.

The KPP partial differential equation on the whole real line,

$$\partial_t h = \frac{1}{2}\partial_{xx} h + \beta(h^2 - h), \tag{14}$$

where $0 \le h(t,x) \le 1$, $h(t,-\infty) = 0$ and $h(t,+\infty) = 1$, describes how a stable phase ($h = 0$ on the left) invades an unstable phase ($h = 1$ on the right). It is well known that it admits travelling wave solutions of the form

$$h(t,x) = h_\mu(x - \mu t), \qquad 0 < h_\mu < 1, \qquad h_\mu(+\infty) = 1, \qquad h_\mu(-\infty) = 0,$$

for any velocity μ greater or equal to $\sqrt{2\beta}$. The travelling wave $x \mapsto h_\mu(x)$ is then solution to

$$\frac{1}{2}h_\mu'' + \mu h_\mu' + \beta(h_\mu^2 - h_\mu) = 0, \quad h_\mu(-\infty) = 0, \quad h_\mu(+\infty) = 1. \tag{15}$$

The solution to (15) is unique up to translation.

There is a large and rich literature devoted to the study of the KPP equation, its traveling wave solutions (and indeed to the convergence towards the traveling wave solutions) going all the way back to the original papers of Kolmogorov et al. [22] and Fisher [13]. To cite only some of the many significant contributions, let us mention: Aronson and Weinberger [4], Fife and McLeod [12], Uchiyama [28], Kametaka [20], as well as some more recent works of Nollen, Roquejoffre and Ryzhik [15, 16]. The KPP equation can also be tackled from a probabilistic point of view thanks to a Feynman-Kac type connection with branching Brownian motion. This approach also has a rich history, the seminal works in this direction being those of McKean [25] and Bramson [8].

In the present work we draw on techniques from both partial differential equations (such as maximum and comparison principles, phase plane analysis) and from probability (martingales, spine decompositions and probability tilting for branching particle systems) to study (D_s) and the behaviour of $K(\infty)$.

Before embarking on the proofs of our results, let us emphasize here that (D_s) really is quite different from (15), although they might appear similar at first glance. In particular, h_μ, surprisingly, does not have the same asymptotic behaviour as ω for large x, in the region where h_μ and ω are close to 1. Indeed, linearising (15) around 1, one gets

$$\frac{1}{2}(1 - \tilde{h}_\mu)'' + \mu(1 - \tilde{h}_\mu)' + \beta(1 - \tilde{h}_\mu) = 0, \quad \text{[linearized]} \tag{16}$$

(a term of order $(1 - \tilde{h}_\mu)^2$ has been neglected) and the general solution to (16) is, for some constants A and B,

$$1 - \tilde{h}_\mu(x) = \begin{cases} Ae^{-\left(\mu - \sqrt{\mu^2 - 2\beta}\right)x} + Be^{-\left(\mu + \sqrt{\mu^2 - 2\beta}\right)x} & \text{for } \mu > \sqrt{2\beta}, \\ (Ax + B)e^{-\sqrt{2\beta}\,x} & \text{for } \mu = \sqrt{2\beta}. \end{cases} \quad \text{[linearized]} \tag{17}$$

For x large, $\tilde{h}_\mu$ is close to h_μ with the meaning that for some constant A and B, $1 - \tilde{h}_\mu \sim 1 - h_\mu$ (see the discussion in Section 2.2.3). Of course, if $A \neq 0$, the term in factor of B is negligible compared to the term in factor of A and the A term alone is an equivalent to $1 - h_\mu$. When solving (15), it turns out that the solution has a non-zero A term and that, therefore,

$$1 - h_\mu(x) \sim \begin{cases} Ae^{-\left(\mu - \sqrt{\mu^2 - 2\beta}\right)x} & \text{for } \mu > \sqrt{2\beta}, \\ Axe^{-\sqrt{2\beta x}} & \text{for } \mu = \sqrt{2\beta}, \end{cases} \tag{18}$$

where A depends on μ.

We now consider equation (D_0) for $\omega(x)$. Of course, the boundary condition of (D_0) is not sufficient to determine a unique solution, and for a range of values of c there exists a solution to

$$0 = \frac{1}{2}v'' + \mu v' + \beta(v^2 - v), \quad v(+\infty) = 1, \quad v(0) = 0, \quad v'(0) = c. \tag{19}$$

(The difference with equation (D_0) for $\omega(x)$ is the added condition $v'(0) = c$. Figure 3 shows several solutions.)

One can then do, as above, a large x analysis of v and, the partial differential equation being the same, one finds again that $1 - v \sim 1 - \tilde{h}_\mu$ (x large) as given in (17) for some c and μ dependent values of A and B. Generically, A is non-zero and $1 - v$ decays as $1 - h_\mu$ in (18) (up to a multiplicative constant; the A is usually different and can even be negative, see Figure 3).

However, for a well chosen value of c (depending on μ), one has $A = 0$, and the asymptotic decay of $1-v$ is given by the B term (that is: it decays much faster). The meaning of Proposition 4 is that $\omega(x)$ is precisely that very special solution to (D_0) that decays unlike all the other ones and unlike the travelling wave h_μ.

The above discussion is made rigorous in the proof of Proposition 4 where we use a phase-plane analysis approach.

It is also interesting to remark that an equation very similar to (D_0) appears in the study [18, 7] of the the extinction probability of a branching Brownian motion with absorption and supercritical drift $\mu > -\sqrt{2\beta}$ (regimes B and C): let $\theta(x) = \mathbb{P}^x[\mathcal{N}_{\text{live}}(\infty) = \emptyset]$ be the extinction probability when the system is started from x; then one has

$$\begin{cases} 0 = \frac{1}{2}\theta'' + \mu\theta' + \beta(\theta^2 - \theta), \quad x \geq 0, \\ \theta(0) = 1, \ \theta(\infty) = 0. \end{cases} \tag{D_1''}$$

Equations (D_0) and (D_1'') differ only by their boundary conditions; however (D_1'') has a unique solution, whereas (D_0) has many.

A possible way to understand the difference is that an asymptotic analysis of $\theta(x)$ for large x similar to (17) yields only one possible exponential decay: $\theta(x) \sim A\exp[-(\mu + \sqrt{\mu^2 + 2\beta})x]$ for some constant A, which means that, up to translations, there is only one solution which does converge to zero at infinity, whereas for $\omega(x)$ there were two possible exponential decays and infinitely many solutions. Otherwise said, if one were to impose $\theta(0) = 1$ and $\theta'(0) = -c$, there would only be one value of c for which θ would converge to zero at infinity.

1.3 A travelling wave result

In regimes A and B ($\mu < \sqrt{2\beta}$) one trivially has $\omega(x) = 0$ since $K(\infty) > 0$ almost surely. What is more interesting in this case is the way that $u(t, x) := \mathbb{P}^x\big[K(t) = 0\big]$ (as defined in (8)) converges to zero: it does so by assuming the shape of the critical travelling wave of the KPP equation. Let us recall quickly the well known facts on this critical travelling wave.

Consider the KPP equation (14) without drift on the whole line with Heaviside initial conditions:

$$\begin{cases} \partial_t h = \frac{1}{2}\partial_{xx}h + \beta(h^2 - h), \\ h(0, x) = 0 \ (\forall x < 0), \qquad h(0, x) = 1 \ (\forall x > 0). \end{cases} \tag{20}$$

It is well known that $h(t, x)$ is the probability that the leftmost particle at time t of a branching Brownian motion started at x is to the right of zero. Furthermore, this probability converges to the critical travelling wave in the following sense:

$$h(t, m_t + x) \to h_*(x) \text{ uniformly in } x \text{ as } t \to \infty$$

with

$$m_t := \sqrt{2\beta}\, t - \frac{3}{2\sqrt{2\beta}} \log t + \text{Cste} \tag{21}$$

and where $h_* := h_{\sqrt{2\beta}}$ is the travelling wave moving at the minimal possible velocity $\sqrt{2\beta}$, see (15). To fix the invariance by translation, we impose the further condition $h_*(0) = 1/2$:

$$\begin{cases} 0 = \frac{1}{2}h_*'' + \sqrt{2\beta}\, h_*' + \beta(h_*^2 - h_*), \\ h_*(-\infty) = 0, \quad h_*(0) = \frac{1}{2}, \quad h_*(+\infty) = 1, \end{cases} \tag{22}$$

and the solution to (22) is now unique.

Adding a drift $+\mu\partial_x h$ to (20) would only shift the solutions by $-\mu t$ and would make (20) very similar to (9): the only difference would be that u is defined on $\mathbb{R}^+$ and h on $\mathbb{R}$. However, as both equations converge quickly to zero around the origin in regimes A and B ($\mu < \sqrt{2\beta}$), this difference turns out to be minimal and one has the following:

Theorem 10. *In regimes A and B ($\mu < \sqrt{2\beta}$), there exists a constant C depending on μ and β such that*

$$u(t, x + m_t - \mu t + C) \to h_*(x) \quad \text{uniformly in } x \text{ as } t \to \infty$$

where m_t is given by (21) *and h_* is the solution to* (22).

It is interesting to compare this result about the behaviour of $u(t,x) = \mathbb{P}^x(K(t) = 0)$ when $\mu < \sqrt{2\beta}$ to the behaviour of the extinction probability $\tilde{u}(t,x) = \mathbb{P}^x(\mathcal{N}_{\text{live}}(t) = \emptyset)$ when $\mu \leq -\sqrt{2\beta}$. It is not hard to see that $\tilde{u}$ satisfies the same equation (9) as u with different boundary conditions, which is 1 minus the boundary condition in (9); namely $\tilde{u}$ solves

$$\begin{cases} \partial_t\tilde{u} = \frac{1}{2}\partial_{xx}\tilde{u} + \mu\partial_x\tilde{u} + \beta(\tilde{u}^2 - \tilde{u}), \\ \tilde{u}(t,0) = 1 \ (\forall t \geq 0), \quad u(0,x) = 0 \ (\forall x > 0), \end{cases} \tag{23}$$

What is particularly striking is that in the critical case, $\mu = -\sqrt{2\beta}$, it is known since Kesten [21] (see also [6]) that to survive up to time t one must start with an initial particle at position $x = ct^{1/3}$. This means that if $\tilde{u}(t, x + \tilde{m}_t)$ converges to some limit front shape then the centering term giving the position of the front $\tilde{m}_t$ has to be of order $ct^{1/3}$. However the convergence of the solution of (23) to a travelling wave is at present an open problem.

2 Proofs

The proofs section are presented mostly in the same order as the results: Section 2.1 contains the proofs to Theorem 1 and Theorem 2 about the structure of the functions $\omega_s(x) = \mathbb{E}^x[s^{K(\infty)}]$ in regime C ($\mu \geq \sqrt{2\beta}$). Section 2.2 contains the proofs of Proposition 4, Theorem 6, Lemma 7, Theorem 8 and Proposition 9 about the different representations of $\omega(x) = \omega_0(x) = \mathbb{P}^x[K(\infty) = 0]$ in regime C ($\mu \geq \sqrt{2\beta}$). Finally, Section 2.3 contains the proof of Theorem 10 about the establishment of a travelling for $u(t,x) = \mathbb{P}^x[K(t) = 0]$ in regimes A and B ($\mu < \sqrt{2\beta}$) and the asymptotic behavior of s_0 established in Proposition 3 is proven last.

2.1 Proof concerning the tail behaviour of $K(\infty)$

In this section we consider exclusively regime C ($\mu \geq \sqrt{2\beta}$) and we focus on the problem of the exponential moments of $K(\infty)$. We first establish some properties of $\omega_s(x) = \mathbb{E}^x[s^{K(\infty)}]$ as defined in (1) and proceed to prove Theorems 2 and Proposition 3. We then prove the asymptotic behaviour (2) to complete the proof of Theorem 1.

The first property we need is that for a given s, the quantity $\omega_s(x)$ is either finite for all $x > 0$ or infinite for all $x > 0$:

Lemma 11. *For a given s,*

$$\big(\exists x > 0 \ : \ \omega_s(x) < +\infty\big) \Leftrightarrow \big(\forall x > 0 \ : \ \omega_s(x) < +\infty\big).$$

Proof. Fix $s > 0$, $x > 0$ and $y > 0$. There is a positive probability, which we note $\epsilon(x, y)$, that the initial particle starting from x reaches position y before any branching or killing happens. Then

$$\omega_s(x) = \mathbb{E}^x[s^{K(\infty)}] \geq \epsilon(x, y)\mathbb{E}^y[s^{K(\infty)}] = \epsilon(x, y)\omega_s(y).$$

Therefore, if $\omega_s(x)$ is finite, then $\omega_s(y)$ is also finite. $\square$

Remark: in the following, we write $\omega_s < \infty$ when the conditions of the lemma are met. Clearly, this is the case when $s \leq 1$. Furthermore, as $s \mapsto \omega_s(x)$ is obviously increasing, if $\omega_{s_0} < \infty$ for some s_0, then $\omega_s < \infty$ for all $s < s_0$.

When $\omega_s < \infty$, it is clear by standard arguments that $\omega_s(x)$ is solution to

$$\begin{cases} 0 = \frac{1}{2}\omega_s'' + \mu\omega_s' + \beta(\omega_s^2 - \omega_s), \\ \omega_s(0) = s. \end{cases} \tag{24}$$

Let us now prove Theorem 2. A slightly more general result is given by the following Lemma:

Lemma 12. *If $\omega_s < \infty$ then, for any $x \geq 0$ and $h \geq 0$,*

$$\omega_s(x + h) = \omega_{\omega_s(h)}(x).$$

Using this lemma, first setting $s = 0$ and then renaming $\omega_0(h)$ as some general $s \in (0, 1)$ gives the first line of Theorem 2. Once we have proved that s_0 exists, similarly setting $s = s_0$ and then renaming $\omega_{s_0}(h)$ as s gives the second line of the Theorem.

Proof. Instead of starting our branching process at position x and killing particles at 0, it is here more convenient to think of the process as started at 0 and particles being absorbed at $-x$. This allows to couple different values of the killing position. In particular, if $\mathcal{H}_x$ designates the particles stopped when they first hit $-x$ and $K_x(\infty)$ is the number of particles in $\mathcal{H}_x$, we have that

$$\omega_s(x + h) = \mathbb{E}\big[\prod_{u \in \mathcal{H}_x} s^{K_h^{(u)}(\infty)}\big] = \mathbb{E}\big[\prod_{u \in \mathcal{H}_x} \omega_s(h)\big] = \mathbb{E}\big[\omega_s(h)^{K_x(\infty)}\big] = \omega_{\omega_s(h)}(x),$$

where $K_h^{(u)}(\infty)$ is the total number of descendent of the particle u which are killed at $-x - h$ (which by translation invariance of the branching Brownian motion and the branching property is an independent copy of $K_h(\infty)$). $\square$

We now have a monotonicity result:

Lemma 13.

1. *If $s < 1$, $x \mapsto \omega_s(x)$ is an increasing function converging to 1.*

2. *If $s > 1$ and $\omega_s < \infty$, $x \mapsto \omega_s(x)$ is a decreasing function converging to 1.*

Proof. Once the increasing/decreasing part is proved, the fact that the limit is 1 is obvious: from its definition, it is clear that $\omega_s < 1$ if $s < 1$ and $\omega_s > 1$ if $s > 1$. Assuming ω_s is increasing or decreasing (depending on s), it must have a limit, and from (24) that limit must be 1.

From its interpretation as the distribution of the all-time minimum of a branching Brownian motion, see (3), it is furthermore clear that $\omega_0 = \omega$ is an increasing function. Then, the coupling provided by Lemma 12 (or more simply Theorem 2) implies that ω_s is an increasing function for all $s < 1$.

Therefore, it only remains to prove that for $s > 1$, ω_s is decreasing when it is finite.

Assume $s > 1$ and $\omega_s < \infty$. We first show that ω_s is monotonous by considering two cases:

- If $\omega_s'(0) > 0$ then, for all $h > 0$ small enough, $\omega_s(h) > s$. But, for x fixed, $s \mapsto \omega_s(x)$ is a strictly increasing function so $\omega_{\omega_s(h)}(x) > \omega_s(x)$. Then by Lemma 12, $\omega_s(x+h) > \omega_s(x)$ for all x and all $h > 0$ small enough: ω_s is increasing.

- If $\omega_s'(0) \le 0$ then, for all $h > 0$ small enough, $\omega_s(h) < s$ because in the limit case $\omega_s'(0) = 0$, one has $\omega_s''(0) < 0$ from (24). Then, as in previous case, $\omega_s(x+h) = \omega_{\omega_s(h)}(x) < \omega_s(x)$ for all x and all $h > 0$ small enough: ω_s is decreasing.

It now remains to rule out the possibility that ω_s is increasing for $s > 1$. Imagine that $s > 1$ and ω_s increases. Then, from (24), $\omega_s''(x) \le -2\beta(\omega_s^2(x) - \omega_s(x)) \le -2\beta(s^2 - s)$ and $\omega_s'(x) \le \omega_s'(0) - 2\beta(s^2 - s)x$, which becomes negative for x large enough, in contradiction with the fact that ω_s increases. So ω_s must decrease for $s > 1$. □

We need now to characterize the values of s for which $\omega_s < \infty$.

Lemma 14. *Assume $s > 1$. If there exists a function v which solves*

$$\begin{cases} 0 = \frac{1}{2}v'' + \mu v' + \beta(v^2 - v), \\ v(0) = s, \qquad v(x) \ge 1 \ (\forall x \ge 0), \end{cases} \tag{D_s'}$$

then $\omega_s < \infty$.

Remark: The converse is obvious: when ω_s is finite, it is one of the solutions to (D_s'). This Lemma allows to define

$$\begin{aligned} s_0 &= \sup\{s \ge 1 : \omega_s < \infty\}, \\ &= \sup\{s \ge 1 : \text{a solution to } (D_s') \text{ exists}\}, \end{aligned} \tag{25}$$

and because $s \mapsto \omega_s(x)$ increases, one has $\omega_s < \infty$ for all $s < s_0$.

Proof. We present two proofs: one probabilistic and one analytical.

Choose $s > 1$ such that (D_s') has a solution v. We introduce the process

$$M_t := \prod_{u \in \mathcal{N}_{\text{live+abs.}}(t)} v\big(X_u(t)\big),$$

where we recall that $\mathcal{N}_{\text{live+abs.}}(t)$ is the set of particles in the branching Brownian motion where particles are frozen at the origin, see Section 1.2.2.

M_t is a positive local martingale and therefore a positive super-martingale which thus converges almost surely to M_∞. Observe that under $\mathbb{P}^x$

$$v(x) = M_0 \ge \mathbb{E}^x(M_t) \ge \mathbb{E}^x(M_\infty).$$

But since for all $t \ge 0$ one has

$$M_t \ge v(0)^{K(t)} = s^{K(t)},$$

we see that $M_\infty \ge s^{K(\infty)}$ and therefore

$$\omega_s(x) = \mathbb{E}^x\big[s^{K(\infty)}\big] \le v(x) < \infty.$$

The same result can be proved analytically through the maximum principle. Let us introduce

$$u_s(t,x) := \mathbb{E}^x[s^{K(t)}], \tag{26}$$

which is clearly solution to

$$\begin{cases} \partial_t u_s = \frac{1}{2}\partial_{xx}u_s + \mu\partial_x u_s + \beta(u_s^2 - u_s), \qquad x \ge 0, \\ u_s(t,0) = s \ (\forall t \ge 0), \qquad u_s(0,x) = 1 \ (\forall x > 0). \end{cases} \tag{27}$$

(Compare to (8).) With v as above, one clearly has $\forall x \geq 0$, $u_s(0,x) \leq v(x)$. Therefore, by the maximum principle we have that

$$u_s(t,x) \leq v(x), \qquad \forall t \geq 0, x \geq 0$$

and as $u_s(t,x) \nearrow \omega_s(x)$ as $t \to \infty$ we see that $\omega_s(x) \leq v(x) < \infty$. $\square$

It is obvious that s_0 defined in (25) depends only on the ratio $\mu/\sqrt{\beta}$ by a simple scaling argument: the branching Brownian motion with drift μ and branching rate β is transformed, when time is scaled by λ and space by $\sqrt{\lambda}$, into a branching Brownian motion with drift $\mu\sqrt{\lambda}$ and branching rate $\beta\lambda$. In particular, $\omega_{s,\beta,\mu}(x) = \omega_{s,\beta\lambda,\mu\sqrt{\lambda}}(\sqrt{\lambda}\,x) = \omega_{s,1,\mu/\sqrt{\beta}}(x/\sqrt{\beta})$ with the obvious new notation. What remains to be shown are the following properties of ω_s: s_0 is finite, ω_{s_0} is finite, $\omega'_{s_0}(0) = 0$ and $s_0 > 1$.

Lemma 15. *$s_0 < \infty$, that is: $K(\infty)$ does not have exponential moments of all orders.*

Proof. For the system started from $x > 0$, consider the following family of events for $n \in \mathbb{N}$:

$$\mathcal{A}_n = \begin{cases} K(\infty) = n, \quad \text{and} \\ \text{for all integers } i \leq n,\ K(i) = i, \quad \text{and} \\ \text{for all integers } i \leq n, \text{ there is at time } i \text{ only one particle alive and it sits in } [x, x+1]. \end{cases}$$

In words, for each $i \in \{0, 1, \ldots, n-1\}$ there is one particle alive at time i, it sits in $[x, x+1]$, and during a time interval one, this particle splits exactly once, one the offspring gets absorbed and the other is again in $[x, x+1]$ at time $i+1$. The one particle alive at time n generates a tree drifting to infinity with no more absorbed particles.

Let $\epsilon_{y,z}\mathrm{d}z$ be the probability that a particle sitting at y has, during a time interval one, exactly one splitting event with one offspring being absorbed and the other one ending up in $\mathrm{d}z$. Define furthermore

$$q = \min_{y \in [x,x+1]} \int_x^{x+1} \epsilon_{y,z} \mathrm{d}z.$$

q is the minimal probability for a particle sitting somewhere in $[x, x+1]$ to have, during a time interval one, exactly one splitting event with one offspring being absorbed and the other one ending up in $[x, x+1]$. It is clear that $q > 0$ and that, furthermore,

$$\mathbb{P}^x(K(\infty) = n) > \mathbb{P}^x(\mathcal{A}_n) > q^n \mathbb{P}^x(K(\infty) = 0).$$

This implies that $\mathbb{E}^x[q^{-K(\infty)}] = \infty$ and that $s_0 \leq 1/q < \infty$. $\square$

Lemma 16. *$\omega_{s_0} < \infty$.*

Proof. If $s_0 = 1$, this is trivial as $\omega_1 = 1$. Assume now $s_0 > 1$ and let us fix $x > 0$. For any $1 < s < s_0$, as ω_s is decreasing, one has $\omega_s(x) < s < s_0$. This implies that (using the monotone convergence Theorem) $\omega_{s_0}(x) = \lim_{s \nearrow s_0} \omega_s(x)$ is finite which entails the result. $\square$

Lemma 17. *$\omega'_{s_0}(0) = 0$.*

Proof. We already know that $\omega'_{s_0}(0) \leq 0$. Assuming $\omega'_{s_0}(0) < 0$, one could continue the function ω_{s_0} to negative arguments using (D'_s) and one could find a $x_0 < 0$ such that $\omega_{s_0}(x_0) > s_0$; then the function $x \mapsto \omega_{s_0}(x_0 + x)$ satisfies (D'_s) with $s > s_0$, which is a contradiction. $\square$

The proof that $s_0 > 1$ for all $\mu \geq \sqrt{2\beta}$ is divided into two steps. First, we show that $s_0 > 1$ in the critical case $\mu = \sqrt{2\beta}$. Then we conclude by proving Proposition 3, which states that s_0 is an increasing function of $\mu/\sqrt{\beta}$.

Lemma 18. *In the critical case $\mu = \sqrt{2\beta}$, for $s > 1$ small enough, there exists solutions to (D'_s), that is $s_0 > 1$.*

Proof. Assume $\mu = \sqrt{2\beta}$. After the change of variables $\ell(x) := e^{\mu x}\big(v(x) - 1\big)$, (D'_s) reads

$$\frac{1}{2}\ell'' + \beta e^{-\mu x}\ell^2 = 0. \tag{28}$$

Let us consider the solution to (28) with $\ell(0) = \ell'(0) = \epsilon$ for some $\epsilon > 0$. We want to prove that $\forall x, \ell(x) > 0$ if ϵ is small enough. Assume otherwise and call $x_0 = \inf\{x \geq 0 : \ell(x) = 0\}$. Then, as $\ell''(x) \leq 0$, we have $\ell(x) \leq \epsilon + x\epsilon$ and thus on $[0, x_0]$ (where $\ell(x) \geq 0$),

$$\ell''(x) \geq -2\beta\epsilon^2(1+x)^2 e^{-\mu x}.$$

We conclude that

$$\ell'(x_0) \geq \epsilon - 2\beta\epsilon^2 \int_0^{x_0} (1+x)^2 e^{-\mu x}\,\mathrm{d}x \geq \epsilon - 2\beta\epsilon^2 \int_0^{\infty} (1+x)^2 e^{-\mu x}\,\mathrm{d}x,$$

which is strictly positive for ϵ small enough. This contradicts the definition of x_0 and thus we have found a solution of (28) such that $\ell(x) > 0$ for all $x \geq 0$. Then $v(x) = 1 + \ell(x)e^{-\mu x}$ is a solution to (D'_s) started from $s = 1 + \epsilon$; in other words $s_0 \geq 1 + \epsilon$ in the $\mu = \sqrt{2\beta}$ case. □

We now proceed to prove that s_0 is a strictly increasing function of $\mu/\sqrt{\beta}$.

Proof of Proposition 3. Let us fix $\mu \geq \sqrt{2\beta_1} > \sqrt{2\beta_2}$. One can easily construct two branching Brownian motions with parameters (μ, β_1) and (μ, β_2) on the same probability space to realise a coupling so that the particles of the second one are a subset of the particles of the first one. It is then clear that for any $s > 1$ one has

$$\omega_{s,\beta_1,\mu}(x) \geq \omega_{s,\beta_2,\mu}(x) \tag{29}$$

(with the obvious extension of notation) so that

$$s_0(\mu/\sqrt{\beta_1}) \leq s_0(\mu/\sqrt{\beta_2}). \tag{30}$$

This already gives non-strict monotonicity and concludes the proof that $s_0 > 1$ for all μ, β with $\mu \geq \sqrt{2\beta}$.

We can now prove that the inequality (30) is strict. Assume otherwise; one would have $\omega_{s_0,\beta_1,\mu}(0) = \omega_{s_0,\beta_2,\mu}(0) = s_0$ (where $s_0 > 1$ would be the common value), $\omega'_{s_0,\beta_1,\mu}(0) = \omega'_{s_0,\beta_2,\mu}(0) = 0$ (from Lemma 17) and, from (24), $\omega''_{s_0,\beta_1,\mu}(0) = -\beta_1(s_0^2 - s_0) < \omega''_{s_0,\beta_2,\mu}(0) = -\beta_2(s_0^2 - s_0)$, which would imply by Taylor expansion that for $x > 0$ small enough $\omega_{s_0,\beta_1,\mu}(x) < \omega_{s_0,\beta_2,\mu}(x)$ in contradiction with (29). □

Finally, the only remaining point to complete the proof of Theorem 1 is the asymptotic behaviour (2), i.e. the assertion that for $x > 0$ fixed

$$p_n(x) := \mathbb{P}^x[K(\infty) = n] \sim \frac{-\omega'_{s_0}(x)}{2s_0^n n^{\frac{3}{2}}\sqrt{\pi\beta(s_0-1)}} \qquad \text{as } n \to \infty. \tag{31}$$

Write $D(z, r)$ for the open disc of the complex plane with center $z \in \mathbb{C}$ and radius r. We extend the definition of $s \mapsto \omega_s(x)$ to $s \in \mathbb{C}$:

$$\omega_s(x) = \mathbb{E}^x[s^{K(\infty)}] = \sum_{n \geq 0} s^n p_n(x), \quad \forall s \in \mathbb{C}. \tag{32}$$

This quantity is analytical on $D(0, s_0)$ because the p_n in (32) are positive and the first singularity on the real axis is at s_0. Furthermore, it is finite on $\overline{D(0, s_0)}$ by uniform convergence because it is finite at s_0.

The arguments we use are extremely close to those used by Maillard in [24]. The key argument, which improves on usual Tauberian theorems, is an application of [14, Corollary VI.1] relying on the analysis of generating functions near their singular points. We need to show that

Lemma 19. *Fix $x > 0$. There exists $r_x > 0$ such that $s \mapsto \omega_s(x)$ is analytical in $V = D(s_0, r_x) \setminus [s_0, \infty)$, and*

$$\partial_s \omega_s(x) \sim \frac{-\omega'_{s_0}(x)}{2\sqrt{\beta(s_0 - s)(s_0^2 - s_0)}} \qquad \text{as } s \to s_0, s \in V, \tag{33}$$

and that

Lemma 20. *Fix $x > 0$. There exists $\epsilon > 0$ such that $s \mapsto \omega_s(x)$ is analytical on $D(0, s_0 + \epsilon) \setminus [s_0, \infty)$.*

Then, applying Corollary VI.1 in [14] to $s \mapsto \partial_s \omega_s(x)$, Lemmas 19 and 20 lead to

$$(n+1)p_{n+1}(x) \sim \frac{-\omega'_{s_0}(x)}{2 s_0^{n+1} n^{\frac{1}{2}} \sqrt{\pi \beta(s_0 - 1)}}, \qquad \text{as } n \to \infty, \tag{34}$$

which obviously implies (31).

Proof of Lemma 19. We know that $\omega'_{s_0}(0) = 0$ and $\omega''_{s_0}(0) = -2\beta(s_0^2 - s_0) < 0$. Since ω_{s_0} solves the KPP traveling wave differential equation, for each $x \geq 0$ we can extend $z \mapsto \omega_{s_0}(x + z)$ analytically on a neighborhood of zero in $\mathbb{C}$ (see e.g. [27]). In particular for $x = 0$ we have the following expansion:

$$\omega_{s_0}(z) = s_0 + \frac{\omega''_{s_0}(0)}{2} z^2 + o(z^2) \quad \text{as } z \to 0. \tag{35}$$

The function ω_{s_0} is analytic and zero is a zero of order two of $\omega_{s_0}(z) - s_0$, by Theorem 10.32 of [27] there exists $r_1 > 0$ and a function ψ analytic and invertible on $D(0, r_1)$ such that

$$\omega_{s_0}(z) = s_0 + \frac{\omega''_{s_0}(0)}{2} \psi(z)^2. \tag{36}$$

This means that

$$z = \psi^{-1}\left(\sqrt{\frac{\omega_{s_0}(z) - s_0}{\omega''_{s_0}(0)/2}}\right) = \psi^{-1}\left(\sqrt{\frac{s_0 - \omega_{s_0}(z)}{\beta(s_0^2 - s_0)}}\right) \tag{37}$$

for any z in $D(0, r_1)$ such that $\omega_{s_0}(z) \notin (s_0, \infty)$ (so that the right-hand side is well defined and is analytic on this domain when using the standard definition of the complex square root).

Recall from Lemma 12 that for any non-negative real x and z one has

$$\omega_{\omega_{s_0}(z)}(x) = \omega_{s_0}(z + x). \tag{38}$$

Replace the z in the right-hand side by its expression (37) and write $\omega_{s_0}(z)$ as s to obtain

$$\omega_s(x) = \omega_{s_0}\left(\psi^{-1}\left(\sqrt{\frac{s_0 - s}{\beta(s_0^2 - s_0)}}\right) + x\right) \tag{39}$$

for $s \in \omega_{s_0}\big([0, r_1)\big) = (s_0 - r_2, s_0]$ for some $r_2 > 0$. But (39) is an equality between analytical functions as long as $s \in D(s_0, r_x) \setminus [s_0, \infty)$ for some $r_x > 0$ small enough (one must have $D(s_0, r_x) \subset \omega_{s_0}\big(D(0, r_1)\big)$ for ψ^{-1} to be analytical, which is possible by the open mapping Theorem, and one must have $\psi^{-1}(\ldots)$ small enough for ω_{s_0} to be also analytical). From the analytical continuation principle, (39) must hold on the whole $D(s_0, r_x) \setminus [s_0, \infty)$ domain. Now differentiate with respect to s to get

$$\partial_s \omega_s(x) = -\frac{(\psi^{-1})'\left(\sqrt{\frac{s_0-s}{\beta(s_0^2-s_0)}}\right)}{2\sqrt{\beta(s_0-s)(s_0^2-s_0)}}\omega'_{s_0}\left(\psi^{-1}\left(\sqrt{\frac{s_0-s}{\beta(s_0^2-s_0)}}\right)+x\right), \tag{40}$$

yielding

$$\partial_s \omega_s(x) \sim -\frac{(\psi^{-1})'(0)}{2\sqrt{\beta(s_0-s)(s_0^2-s_0)}}\omega'_{s_0}(x) \quad \text{as } s \to s_0 \text{ in } D(s_0, r_x) \setminus [s_0, \infty). \tag{41}$$

A straightforward computation shows that $(\psi^{-1})'(0) = 1$, which concludes the proof. □

Proof of Lemma 20. $\omega_s(x)$ is already analytical on $D(0, s_0)$. To prove the Lemma it is sufficient to show that it can be analytically extended around any point $s \in \partial D(0, s_0) \setminus \{s_0\}$. Indeed, by the finite covering property of compacts one can then show analycity on a open containing the compact $\partial D(0, s_0) \setminus D(s_0, r_x/2)$ with r_x defined in Lemma 19, and then we conclude with the help of Lemma 19.

So it now remains to see why $s \mapsto \omega_s(x)$ can be analytically extended to neighboorhoods of any $s \neq s_0$ with $|s| = s_0$. This is essentially the content of Lemma 6.2 in [24]. As in Maillard, we define

$$a(s) := \omega'_s(0), \tag{42}$$

where we recall that the prime is a derivatice with respect to x. We first show analycity of $a(s)$ on $D(0, s_0)$ by writing an integral representation of $a(s)$: multiply (D_s) by $\exp\left[(\mu - \sqrt{\mu^2 + 2\beta})x\right]$ and integrate on $x \in [0, \infty)$. Integrate several times by part to get rid of the derivatives of ω_s; one is left with

$$a(s) = \left(\mu - \sqrt{\mu^2 + 2\beta}\right)s + 2\beta \int_0^\infty \mathrm{d}x\, \omega_s(x)^2 e^{\left(\mu - \sqrt{\mu^2+2\beta}\right)x}. \tag{43}$$

For any $x \geq 0$ and $s \in \overline{D(0, s_0)}$ one has $|\omega_s(x)| \leq \omega_{s_0}(x) \leq s_0$. This implies that the convergence for x close to infinity of the integral in (43) is uniform on the disk $s \in \overline{D(0, s_0)}$. As $s \mapsto \omega_s(x)$ is analytical on $D(0, s_0)$, this is sufficient to ensure that $s \mapsto a(s)$ is also analytical on $D(0, s_0)$. Furthermore, notice that the series (32) defining $\omega_s(x)$ converges uniformly on $s \in \overline{D(0, s_0)}$ because we know it converges absolutely (all the $p_n(x)$ are non-negative) at $s = s_0$. This implies that $s \mapsto \omega_s(x)$ is continuous on $\overline{D(0, s_0)}$ and, from the expression (43), so is $s \mapsto a(s)$ (by dominated convergence Theorem since $|\omega_s(x)| \leq s_0$ on the closed disc).

We proceed to show that $a(s)$ can be extended analytically around any point $s \neq s_0$ with $|s| = s_0$ and show that the property extends to $\omega_s(x)$.

In Lemma 12 we showed for any $s \in [0, s_0]$ and any $x \geq 0$ and $h \geq 0$ one had

$$\omega_s(x + h) = \omega_{\omega_s(h)}(x). \tag{44}$$

One can check that the proof of Lemma 12 extends to complex s so that (44) remains valid for any $s \in C$ such that $\omega_s(x)$ is finite.

For fixed (complex) s, by deriving (44) with respect to h and then setting $h = 0$, one gets

$$\omega'_s(x) = a(s)\partial_s \omega_s(x). \tag{45}$$

Derive again with respect to x, and then set $x = 0$:

$$\omega''_s(0) = a(s)\partial_s a(s), \tag{46}$$

so that the differential equation (24) on $x \mapsto \omega_s(x)$ applied at $x = 0$ gives

$$0 = \frac{1}{2} a(s) \partial_s a(s) + \mu a(s) + \beta(s^2 - s), \tag{47}$$

which is equation (3.5) in [24]. This equation is valid for all $s \in D(0, s_0)$.

We now use the Fact 5.2 in [24], which we reproduce here with some change of notation for clarity:

> *Fact 5.2 in [24]* Let H be a region in $\mathbb{C}$ and $s \mapsto \phi(s)$ analytic in H. Let G be a region in $\mathbb{C}^2$ such that $(\phi(s), s) \in G$ for each $s \in H$ and suppose that there exists an analytic function $f : G \to \mathbb{C}$ such that
>
> $$\phi'(s) = f\big(\phi(s), s\big), \quad \forall s \in H.$$
>
> Let $s^* \in \partial H$. Suppose $\phi(s)$ is continuous at s^* and that $(\phi(s^*), s^*) \in G$. Then s^* is a regular point of $\phi(s)$ i.e. $\phi(s)$ admits an analytic continuation at s^*.

We apply to our case with $\phi = a$, $H = D(0, s_0) \setminus \{1\}$ and $G = \mathbb{C}^* \times \mathbb{C}$. From (47), the only candidate values of s in $D(0, s_0)$ such that $a(s) = 0$ are 0 and 1, and we know that $a(0) > 0$, so the condition "$(a(s), s) \in G$ for each $s \in H$" is verified. The function $f(a, s)$ is obtained from (47): $f(a, s) = -2\mu + 2(s - s^2)/a$, and is obviously analytical on G; we have already shown that $a(s)$ is continuous on $\overline{D(0, s_0)}$. Therefore, for any point $s^* \in \partial D(0, s_0)$ such that $a(s^*) \neq 0$ (because we want $(a(s^*), s^*) \in G$), the function $a(s)$ admits an analytic continuation at s^*. We know that $a(s_0) = 0$, and we prove now that one has $a(s^*) \neq 0$ for any $s^* \in \partial D(0, s_0) \setminus \{s_0\}$, which will conclude the proof that $a(s)$ can be analytically continued around any point in $\partial D(0, s_0) \setminus \{s_0\}$. From (32) one can write $a(s)$ as a series:

$$a(s) = \sum_n p'_n(0) s^n. \tag{48}$$

We know that $p'_1(0) \leq 0$ (because $p_1(0) = 1$ and $p_1(x > 0) < 1$) and that for $n \neq 1$, $p'_n(0) \geq 0$ (because $p_n(0) = 0$ and $p_n(x > 0) > 0$). Since $a(s_0) = 0$, we write

$$\sum_{n \neq 1} p'_n(0) s_0^n = -p'_1(0) s_0. \tag{49}$$

All the terms on the left hand side are non-negative and infinitely many of them are non-zero since (47) does not have polynomial solutions. Thus, for any $s^* \in \partial D(0, s_0) \setminus \{s_0\}$ one has

$$\Big| \sum_{n \neq 1} p'_n(0) (s^*)^n \Big| < \sum_{n \neq 1} p'_n(0) s_0^n. \tag{50}$$

In particular, $a(s^*) \neq 0$ because it is the sum of two terms (the $\sum_{n \neq 1}$ and the term $n = 1$) with different moduli and $s \mapsto a(s)$ can be extended analytically around s^*.

We now show how the analycity of $a(s)$ translates into analycity of $\omega_s(x)$. First derive (44) again but this time with respect to x, then set $x = 0$, and rename h into x to obtain

$$\omega'_s(x) = a\big[\omega_s(x)\big] = a(s) \partial_s \omega_s(x), \tag{51}$$

where we used (45) for the second equality.

For each given $s^* \in \partial D(0, s_0) \setminus \{s_0\}$ we consider a neibourhood V of s^* where $a(s)$ is analytical and we apply again Fact 5.2 in [24] to prove that $s \mapsto \omega_s(x)$ is also analytical around s^*. This time, we take $\phi(s) = \omega_s(x)$ and $f(\omega, s) = a(\omega)/a(s)$ from (51). We pick $H = D(0, s_0) \setminus \{1\}$ and $G = \overline{D(0, s_0)} \times (H \cup V)$. For any $s \in \overline{D(0, s_0)} \setminus \{s_0\}$ one has $|\omega_s(x)| < \omega_{s_0}(x) \leq s_0$ so that the condition "$(\phi(s), s) \in G$ for each $s \in H$" is satisfied. We have already shown that $s \mapsto \omega_s(x)$ is continuous on $\overline{D(0, s_0)}$, so we conclude that s^* is a regular point of $s \mapsto \omega_s(x)$.

$\square$

2.2 Proofs concerning $\omega(x)$

In this section we consider exclusively regime C ($\mu \geq \sqrt{2\beta}$) and prove our results concerning the properties of ω and ω_s.

2.2.1 Proof of Theorem 6

We need to prove that ω is the maximal solution remaining below 1 of the differential equation (D_0). This is an elementary application of the maximum principle again. Suppose that v is any solution of (D_0) which stays below 1. Since v is a *standing wave* solution of (9), that is $\tilde{u}(t,x) = v(x)$ for all $t \geq 0$ is a solution of

$$\partial_t \tilde{u} = \frac{1}{2}\partial_{xx}\tilde{u} + \mu\partial_x\tilde{u} + \beta(\tilde{u}^2 - \tilde{u}),$$

and since $v(x) < 1$ for all $x \geq 0$ we have that

$$v(x) \leq u(t,x), \qquad \forall t \geq 0, \forall x \geq 0$$

where u is the solution of (9). As for each x we know that $t \mapsto u(t,x) \searrow \omega(x)$ we conclude that $\omega(x) \geq v(x)$ and therefore ω is the maximal solution of (D_0) bounded by 1. The same argument is easily generalized to the case of an arbitrary value of $s \in [0, s_0]$.

2.2.2 Proof of the martingale representation, Theorem 8

We start by proving Lemma 7, i.e. that the martingale $(Z_{\text{live+abs.}}(t), t \geq 0)$ converges $\mathbb{P}$-almost surely and in L^1 to $K(\infty)$ and therefore that $\mathbb{E}^x[K(\infty)] = e^{-rx}$.

Proof of Lemma 7. Recall that by (10)

$$Z_{\text{live+abs.}}(t) := \sum_{u \in \mathcal{N}_{\text{live+abs.}}(t)} e^{-rX_u(t)}, \qquad Z_{\text{all}}(t) := \sum_{u \in \mathcal{N}_{\text{all}}(t)} e^{-rX_u(t)}, \tag{52}$$

are positive martingales which therefore converge $\mathbb{P}$-almost surely to their respective limits $Z_{\text{live+abs.}}$ and Z_{all}. Furthermore, as $Z_{\text{all}}(t)$ is the usual additive martingale with parameter $r \geq \sqrt{2\beta}$, one has $Z_{\text{all}} = 0$. As the bounds

$$K(t) \leq Z_{\text{live+abs.}}(t) \leq K(t) + Z_{\text{all}}(t)$$

always hold, it is clear that $Z_{\text{live+abs.}} = K(\infty)$. The only thing left is to show that the convergence also holds in L^1.

We start by recalling the description of the measure $\mathbb{Q}^x|_{\mathcal{F}_t}$ which is defined by

$$\left.\frac{\mathrm{d}\mathbb{Q}^x}{\mathrm{d}\mathbb{P}^x}\right|_{\mathcal{F}_t} = \frac{Z_{\text{live+abs.}}(t)}{Z_{\text{live+abs.}}(0)}.$$

Standard arguments (see [17]) allow us to conclude that under $\mathbb{Q}^x$ the process behaves as follows: for $t \geq 0$, there is a distinguished line of descent (the *spine*) denoted $\xi(t) \in \mathcal{N}_{\text{live+abs.}}(t)$. Under $\mathbb{Q}^x$ the particle ξ moves according to a Brownian motion with drift $-\sqrt{\mu^2 - 2\beta}$ and therefore almost surely hits 0 in finite time; we call $\tau_\xi = \inf\{t \geq 0 : X_{\xi(t)}(t) = 0\}$ the time at which it reaches 0. For $t < \tau_\xi$, the spine branches at rate 2β creating non-spine particles which start new independent branching Brownian motion behaving according to the usual $\mathbb{P}$ law. After τ_ξ, the spine particle is frozen at zero (no motion, no branching). Observe that $\mathbb{Q}^x$ is actually the projection of the measure just described since under $\mathbb{Q}^x$ we do not know which is the spine particle ξ.

To prove the L^1 convergence of $Z_{\text{live+abs.}}(t)$ towards its limit $Z_{\text{live+abs.}} := \lim_t Z_{\text{live+abs.}}(t)$, it is sufficient to show that

$$\mathbb{Q}^x(Z_{\text{live+abs.}} < \infty) = 1.$$

As the time τ_ξ at which the spine is absorbed at 0 is $\mathbb{Q}$-almost surely finite, there are only finitely many branching events from the spine $\mathbb{Q}$-almost surely as well. At each of these events, a non-spine particle u starts its own independent $\mathbb{P}$ branching Brownian motion and we call $K_u(\infty)$ the total number of particles frozen at 0 that are descended from u. Let us also call $Z^{(u)}_{\text{live+abs.}}$ the analogue of the limit $Z_{\text{live+abs.}}$ (but we sum only on particles descended from u) and $Z^{(u)}_{\text{all}}$ is the same as $Z^{(u)}_{\text{live+abs.}}$ but without any absorption or freezing at 0. It is clear as above that

$$K_u(\infty) \le Z^{(u)}_{\text{live+abs.}} \le K_u(\infty) + Z^{(u)}_{\text{all}}.$$

and that $Z^{(u)}_{\text{all}} = 0$. We conclude that $Z^{(u)}_{\text{live+abs.}} = K_u(\infty) < \infty$, $\mathbb{Q}$-almost surely, and finally

$$Z_{\text{live+abs.}} = K(\infty) < \infty \quad \mathbb{Q}\text{-almost surely.}$$

Observe that since $K(\infty) \ge 1$ almost surely under $\mathbb{Q}$, we have $\mathbb{Q} \sim \mathbb{P}$. Thus we know that under $\mathbb{P}$ $Z_{\text{live+abs.}}(t) \to Z_{\text{live+abs.}} = K(\infty)$ in L^1. Hence, $\mathbb{E}^x(K(\infty)) = Z_{\text{live+abs.}}(0) = e^{-rx}$. □

We now move to the proof of Theorem 8.

Lemma 21. *Recall $\omega(x) = \mathbb{P}^x(K(\infty) = 0)$ and $\omega_s(x) := \mathbb{E}^x[s^{K(\infty)}]$ for $0 < s \le s_0$ as usual. Then*

$$1 - \omega(x) = \mathbb{Q}^x\Big[\frac{1}{K(\infty)}\Big]e^{-rx}$$

and

$$1 - \omega_s(x) = \mathbb{Q}^x\Big[\frac{1 - s^{K(\infty)}}{K(\infty)}\Big]e^{-rx}.$$

Proof. As $K(\infty) > 0$ $\mathbb{Q}^x$-almost surely, it is sufficient to prove the second assertion. Using that $K(\infty) = Z_{\text{live+abs.}}$ $\mathbb{P}^x$-almost surely,

$$\begin{aligned}
1 - \omega_s(x) &= \mathbb{P}^x[1 - s^{K(\infty)}] = \mathbb{P}^x[(1 - s^{K(\infty)})\mathbb{1}_{\{Z_{\text{live+abs.}}>0\}}] \\
&= \mathbb{P}^x\big[(1 - s^{K(\infty)})\frac{Z_{\text{live+abs.}}(0)}{Z_{\text{live+abs.}}}\frac{Z_{\text{live+abs.}}}{Z_{\text{live+abs.}}(0)}\mathbb{1}_{\{Z_{\text{live+abs.}}>0\}}\big] \\
&= \mathbb{Q}^x\big[\frac{Z_{\text{live+abs.}}(0)}{Z_{\text{live+abs.}}}(1 - s^{K(\infty)})\mathbb{1}_{\{Z_{\text{live+abs.}}>0\}}\big] \\
&= \mathbb{Q}^x\big[\frac{1 - s^{K(\infty)}}{K(\infty)}\big]e^{-rx}.
\end{aligned}$$

□

Since we already know from Proposition 4 (its proof is analytical and is independent of the present discussion) that $(1 - \omega_s(x))e^{rx}$ tends to a constant $B > 0$, it is now clear that the $\mathbb{Q}^x$ expectations in Lemma 21 also converge to B as $x \to \infty$. However, we are now going to define $\mathbb{Q}^\infty$ as the law of the process under which we can couple all the $\mathbb{Q}^x$ together and interpret the limit constant B as the expectation of a limit variable under $\mathbb{Q}^\infty$. Loosely speaking, we want $\mathbb{Q}^\infty$ to be the law of the process where the spine particle starts at $x = +\infty$ before drifting to 0. In fact it is easier to reverse time and have the spine start at 0 and drift to $+\infty$.

We start with the following Lemma:

Lemma 22. *Let $(Y(t), t \geq 0)$ be a Brownian motion with drift $+\sqrt{\mu^2 - 2\beta}$, started from 0 conditioned to never hit 0. Otherwise said, Y is solution of the following stochastic differential equation*

$$\begin{cases} \mathrm{d}Y(t) = \mathrm{d}B_t + \sqrt{\mu^2 - 2\beta}\coth\big(\sqrt{\mu^2-2\beta}\,Y(t)\big)\mathrm{d}t & \text{if } \mu > \sqrt{2\beta} \\ \mathrm{d}Y(t) = \mathrm{d}B_t + \frac{1}{Y(t)}\mathrm{d}t \qquad \text{if } \mu = \sqrt{2\beta}. \end{cases}$$

Let (t_i) be a Poisson point process on $\mathbb{R}^+$ with intensity 2β. For each $i \geq 0$ start a branching Brownian motion with law $\mathbb{P}^{Y(t_i)}$ and call $\tilde{K}_i$ the total number of absorbed particles at 0 for this process.

Fix $x > 0$, then the distribution of the variable $K(\infty)$ under $\mathbb{Q}^x$ is the same as that of

$$K^x(\infty) := 1 + \sum_{i : t_i \leq \tau_x} \tilde{K}_i$$

under $\mathbb{Q}^\infty$ where $\tau_x := \sup_{t \geq 0}\{Y(t) = x\}$.

This result should be clear once it is realized that the process Y is the reversed path of the spine ξ.

Proof. We only treat the case $\mu > \sqrt{2\beta}$ since the zero-drift case is similar. The only thing we need to prove here is that if $(\xi(t), t \leq \tau_\xi)$ is a Brownian motion with drift $-\sqrt{\mu^2 - 2\beta}$ started from x and stopped at time $\tau_\xi := \inf\{t : \xi(t) = 0\}$, then

$$\{(\xi(t), t \leq \tau_\xi), \tau_\xi\} \stackrel{\mathcal{L}}{=} \{(Y(\tau_x - t), t \leq \tau_x), \tau_x\}.$$

This follows for instance from [3]. $\square$

The upshot of Lemma 22 is that we can now construct the variables $K(\infty)$ under $\mathbb{Q}^x$ for all values of x simultaneously. We write $\mathbb{Q}^\infty$ for the joint law of the variables $((Y(t), t \geq 0), (t_i)_{i \in \mathbb{N}}, \tilde{K}_i)$ described above. Then under $\mathbb{Q}^\infty$, clearly $(K^x(\infty), x \geq 0)$ is an increasing process in x. We call $K^\infty(\infty)$ its limit which is also the total number of particles absorbed at 0 under $\mathbb{Q}^\infty$.

Lemma 23. *We have that $K^\infty(\infty) < \infty$ $\mathbb{Q}^\infty$-almost surely.*

Proof. We start with the $\mu > \sqrt{2\beta}$ case. First observe that for $\epsilon > 0$ fixed, there exists almost surely a random $i_0 \in \mathbb{N}$ such that

$$\forall i \geq i_0, \quad Y(t_i) \geq \Big(\frac{\sqrt{\mu^2 - 2\beta}}{2\beta} - \epsilon\Big) i.$$

This simply comes form the fact that $Y(t)/t \to \sqrt{\mu^2 - 2\beta}$ and $t_i / i \to (2\beta)^{-1}$ almost surely. Now, $1 - \omega(x) \leq e^{-(\mu + \sqrt{\mu^2 - 2\beta})x}$ because under $\mathbb{Q}^x$ we have that $Z_{\text{live+abs.}} \geq 1$ almost surely and therefore $\mathbb{Q}^x(1/Z_{\text{live+abs.}}) \leq 1$. Hence, for any $i \geq i_0$ we have that

$$\mathbb{Q}^\infty(\tilde{K}_i > 0) \leq 1 - \omega\Big(\big(\frac{\sqrt{\mu^2 - 2\beta}}{2\beta} - \epsilon\big)i\Big) \leq e^{-ci}$$

for some positive constant c. Thus a straightforward application of Borel-Cantelli Lemma shows that almost surely, there exists $j_0 \in \mathbb{N}$ such that $\forall i \geq j_0, \tilde{K}_i = 0$, which yields the desired result. The zero-drift case is similar. One just needs to start the argument by observing that for $\epsilon > 0$ fixed, there exists almost surely a random $i_0 \in \mathbb{N}$ such that

$$\forall i \geq i_0, \quad Y(t_i) \geq c i^{1/2 - \epsilon}$$

where c is a constant. The proof then follows as before. $\square$

The following Lemma completes the proof of Theorem 8.

Lemma 24. *For $s \in [0, s_0]$ small enough, we have that*

$$\mathbb{Q}^x\Big(\frac{s^{K(\infty)}}{K(\infty)}\Big) \to \mathbb{Q}^\infty\Big(\frac{s^{K^\infty(\infty)}}{K^\infty(\infty)}\Big). \tag{53}$$

Proof. The monotone convergence Theorem applies when $s \leq 1$ so we suppose $1 < s \leq s_0$. Observe that the map $t \mapsto s^t/t$ is decreasing on $[1, 1/\log s]$ and increasing on $[1/\log s, \infty)$. Thus we write

$$\begin{aligned}
\mathbb{Q}^x\Big(\frac{s^{K(\infty)}}{K(\infty)}\Big) &= \mathbb{Q}^\infty\Big(\frac{s^{K^x(\infty)}}{K^x(\infty)}\Big) \\
&= \mathbb{Q}^\infty\Big(\frac{s^{K^x(\infty)}}{K^x(\infty)}; K^x(\infty) \leq 1/\log s\Big) + \mathbb{Q}^\infty\Big(\frac{s^{K^x(\infty)}}{K^x(\infty)}; K^x(\infty) \geq 1/\log s\Big) \\
&\to \mathbb{Q}^\infty\Big(\frac{s^{K^\infty(\infty)}}{K^\infty(\infty)}; K^\infty(\infty) \leq 1/\log s\Big) + \mathbb{Q}^\infty\Big(\frac{s^{K^\infty(\infty)}}{K^\infty(\infty)}; K^\infty(\infty) \geq 1/\log s\Big) \\
&= \mathbb{Q}^\infty\Big(\frac{s^{K^\infty(\infty)}}{K^\infty(\infty)}\Big)
\end{aligned}$$

where the first convergence comes from the dominated convergence Theorem and the second from the monotone convergence Theorem.

□

2.2.3 Proof of Proposition 4

We consider here all the solutions $x \mapsto v(x)$ to (D_0) that remains in $[0, 1)$. By Cauchy's theorem, a solution to (D_0) is entirely determined once the derivative at the origin is given.

Let r and $R < r$ be the two roots of the polynomial $\frac{1}{2}X^2 - \mu X + \beta$:

$$r = \mu + \sqrt{\mu^2 - 2\beta}, \qquad R = \mu - \sqrt{\mu^2 - 2\beta}.$$

(See also (7).) From the general theory of differential equations, one has

Lemma 25. *Let v be a solution to*

$$0 = \frac{1}{2}v'' + \mu v' + \beta(v^2 - v) \tag{54}$$

such that $v(x)$ converges to 1 as $x \to \infty$. Then, for some non-zero constant A or B,

- *if $\mu > \sqrt{2\beta}$ one has either $1 - v(x) \sim Ae^{-Rx}$ or $1 - v(x) \sim Be^{-rx}$ as $x \to \infty$.*
- *If $\mu = \sqrt{2\beta} = r = R$ one has either $1 - v(x) \sim Axe^{-\mu x}$ or $1 - v(x) \sim Be^{-\mu x}$ as $x \to \infty$.*

Furthermore, up to invariance by translation, there are exactly two solutions which converges to 1 in the fast way (as Be^{-rx}); one of them approaching 1 by above and the other from below.

This lemma simply tells that the solutions to the non-linear equation (54) behave around $v = 1$ as the solutions to the equation linearised around 1. We already used implicitly this result in Section 1.2.4.

Proof. This follows from a result of Hartman [19] who shows that if

$$\dot{X} = \Gamma X + F(X) \tag{55}$$

is a non-linear differential system of dimension 2 with Γ a hyperbolic (eigenvalues have a non-zero real part) matrix and F is C^1 with $F(x) = o(|x|)$ as $x \to 0$ (so 0 is a critical point) , then there exists a C^1 diffeomorphism ϕ with derivative the identity at the origin such that $U(t) = \phi(X(t))$ solves

$$\dot{U} = \Gamma U. \tag{56}$$

Otherwise said the solutions of the linearized system and the solutions of the non-linear system are (locally around 0) in one-to-one correspondence through ϕ.

We apply this result to the following system where $v = 1 - u$ is a solution of (54)

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}; \quad \dot{X}(t) = \begin{pmatrix} u'(t) \\ u''(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ -2\mu y(t) - 2\beta\big[x(t) - x^2(t)\big] \end{pmatrix}, \tag{57}$$

which has a critical point at $(x, y) = (0, 0)$. In this case

$$\Gamma = \begin{pmatrix} 0 & 1 \\ -2\beta & -2\mu \end{pmatrix} \tag{58}$$

with eigenvalues $-r = -\mu - \sqrt{\mu^2 - 2\beta}$ and $-R = -\mu + \sqrt{\mu^2 - 2\beta}$ (for simplicity we only consider the case $r \neq R$ here) and corresponding eigenvectors $\begin{pmatrix} 1 \\ -r \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -R \end{pmatrix}$. The solutions of $\dot{U} = \Gamma U$ are of the form

$$U(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = Be^{-rt} \begin{pmatrix} 1 \\ -r \end{pmatrix} + Ae^{-Rt} \begin{pmatrix} 1 \\ -R \end{pmatrix}. \tag{59}$$

Thus the only solutions such that $|U(t)| \sim ce^{-rt}$ for some constant c are those such that $A = 0$. If $B > 0$ then u_1 approaches by above, if $B < 0$ u_1 approaches by below. Hartman's Theorem tells us that there exists

$$\phi(X) = X + f(X), \quad f(X) = o(|X|) \text{ when } x \to 0 \tag{60}$$

such that the solutions $X(t)$ of the non-linear system are locally

$$X(t) = \phi^{-1}(U(t)). \tag{61}$$

Thus, (after a shift in the argument, replacing x by $x + \ln|B|/r$) there is exactly one solution X to the non linear system such that $|X(t)|e^{rt}$ has a non degenerate limit and such that $x(t)$, the first coordinate of $X(t)$, is eventually positive (resp. eventually negative). $\square$

Let $s < 1$ be such that there is a solution v of (54) with $v(0) = s$, $1 - v(x) \sim ce^{-rx}$, and $v(x) < 1$ for all $x \geq 0$ (we now know that such an s exists). Then $\omega_s(x)$ being the maximal solution of (D_0) that starts from s and stays below 1, we must have

$$\omega_s(x) \geq v(x), \forall x \geq 0.$$

Since we also know that the only two possibilities for the asymptotic behavior of ω_s are that either $\omega_s(x)e^{rx} \to c$ or $\omega_s(x)e^{Rx} \to c$ we conclude that it is the former that holds. The same argument apply for $\omega_s(x)$ for any $s \leq s_0$ and in the critical case. This concludes the proof of Proposition 4 for $\omega(x)$.

2.2.4 Proof of Proposition 9

We consider the series $\Phi(z) = \sum_{n\geq 1} a_n z^n$ defined in (12) with the coefficients a_n defined in (11). The function $z \mapsto \Phi(z)$ is a well defined object because, by induction on (11) one has easily $0 < a_n \leq 1$ and, therefore, $\mathcal{R} \geq 1$. It is then very easy to check by direct substitution that for any $B \in \mathbb{R}$, the function

$$x \mapsto v(x) = 1 - \Phi(Be^{-rx}) \quad \text{for } x \text{ such that } |B|e^{-rx} < \mathcal{R}, \tag{62}$$

is solution to the partial differential equation $\frac{1}{2}v'' + \mu v' + \beta(v^2 - v) = 0$ which appears in (D_0) (when discussing ω) and in (24) (when discussing ω_s). Recall that $r = \mu + \sqrt{\mu^2 - 2\beta}$ is the larger root of $\frac{1}{2}X^2 - \mu X + \beta$.

As the coefficients a_n are positive, $\Phi(z)$ is non-negative and increasing for $z \geq 0$. As $a_1 = 1$ and $a_2 > 0$, it is easy to find a $0 < z_0 < 1 \leq \mathcal{R}$ such that $\Phi(z_0) > a_1 z_0 + a_2 z_0^2 > 1$. This implies that there must exist a $B_0 \in (0, \mathcal{R})$ (smaller than z_0) such that $\Phi(B_0) = 1$. With $B = B_0$, the function $v(x)$ in (62) is smaller than 1 and converges to 1 for large x as e^{-rx}. Using Proposition 4, this implies that $v(x) = \omega(x) = 1 - \Phi(B_0 e^{-rx})$.

Recall by Theorem 2 that ω_s for $s < 1$ is simply equal to ω correctly shifted to have $\omega_s(0) = s$. This implies that, for $s < 1$, $\omega_s(x) = 1 - \Phi(B_s e^{-rx})$ where $B_s \in (0, B_0]$ is such that $\Phi(B_s) = 1 - s$.

The case $s = 1$ is trivial, we now turn to $s > 1$. As for $s = 0$, we have the following points:

- for $s > 1$, ω_s is the smallest solution to (24) that remains above 1 (Theorem 6).
- By Lemma 25, there is exactly one solution to (24) which remains above 1 and decays to 1 as e^{-rx}. Because of the previous point, this solution must be ω_s.

Now consider $\Phi(z)$ for negative arguments. Because $\Phi(0) = 0$ and $\Phi'(0) = 1$, there must exist $B \in (-\mathcal{R}, 0)$ such that Φ is negative on $[B, 0)$. Then, the function $x \mapsto 1 - \Phi(Be^{-rx})$ is solution to (24) for $s = 1 - \Phi(B) > 1$, remains above 1 for $x \geq 0$ and converges to 1 as e^{-rx}. Therefore, it must be ω_s for that particular s.

But all the functions ω_s for $1 < s \leq s_0$ are related through Theorem 2: they are all shifted versions of ω_{s_0}. Therefore, for any $s \in (1, s_0]$, one has $\omega_s(x) = 1 - \Phi(B_s e^{-rx})$ for a well chosen negative B_s (which represents the shift), at least for values of x sufficiently large to have $|B_s|e^{-rx} < \mathcal{R}$.

2.3 Proof of Theorem 10

We assume to be in regime A or B ($\mu < \sqrt{2\beta}$) and we want to show how $u(t, x) = P^x(K(t) = 0)$ converges to a KPP travelling wave.

The proof is essentially analytic and relies on Bramson's result [9] and the maximum principle. The key step is to compare $u(t, x)$ to a new function $v^T : [T, +\infty) \times \mathbb{R} \mapsto \mathbb{R}$ (where $T \geq 0$ is a parameter) where $v^T(t, x)$ is defined as the probability, in the standard branching Brownian motion (without absorption nor stopping) with drift μ starting from x, that no particles are present in the negative half-line between times $t - T$ and t. In symbols

$$v^T(t, x) := \mathbb{P}^x\big(\forall r \in [t - T, t], \forall u \in \mathcal{N}_{\text{all}}(r) : X_u(r) > 0\big), \tag{63}$$

where we recall that $\mathcal{N}_{\text{all}}(s)$ is the population of particles at time s in a branching Brownian motion without absorption or stopping.

The advantage of v^T is that since it is defined on $\mathcal{N}_{\text{all}}$ it satisfies a KPP equation on the whole real line:

$$\begin{cases} \partial_t v^T = \frac{1}{2}\partial_{xx} v^T + \mu \partial_x v^T + \beta\big((v^T)^2 - v^T\big), & (t, x) \in [T, +\infty) \times \mathbb{R} \\ v^T(T, x) = u(T, x), \quad \text{for } x \geq 0, \\ v^T(T, x) = 0, \quad \text{for } x < 0. \end{cases}$$

Otherwise said the function $\tilde{v}^T(t,x) = v^T(T+t,x)$ solves the KPP equation on the whole line with initial condition $\tilde{v}^T(0,x) = u(T,x)\mathbb{1}_{\{x>0\}}$. Since for $T > 0$ fixed, $1 - u(T,x)$ goes to 0 as $x \to \infty$ with a super exponential decay (like the tail of a Gaussian), Bramson's convergence Theorem [9, Theorem A], ensures that there exists a constant $\tilde{C}_T \in \mathbb{R}$ such that we have

$$\|\tilde{v}^T(t, \cdot + m_t - \mu t + \tilde{C}_T) - h_*(\cdot)\|_\infty \to 0 \text{ as } t \to \infty, \tag{64}$$

where Bramson's displacement m_t is given in (21). The value $\tilde{C}_T$ depends on T because for different T we plug different initial conditions in the KPP equation .

Since $m_t - m_{t-T} \to \sqrt{2\beta}\,T$ when $t \to \infty$, one obtains taking $C_T = \tilde{C}_T - \sqrt{2\beta}\,T + \mu T$:

$$\|v^T(t, \cdot + m_t - \mu t + C_T) - h_*(\cdot)\|_\infty \to 0 \text{ as } t \to \infty. \tag{65}$$

Therefore we only need to show that for t large enough, $u(t,x)$ is close to $v^T(t,x)$:

Lemma 26.

$$\|v^T(\cdot,\cdot) - u(\cdot,\cdot)\|_\infty \to 0 \text{ as } T \to \infty. \tag{66}$$

In addition, there exists $C \in \mathbb{R}$ such that $C_T \to C$ as $T \to \infty$.

Indeed, assuming that Lemma 26 holds, we can conclude the

Proof of Theorem 10. Fix $\epsilon > 0$. Using (65) and (66), choose T large enough that $\|v^T(\cdot,\cdot) - u(\cdot,\cdot)\|_\infty < \epsilon$ and then choose t large enough so that $\|v^T(t, \cdot + m_t - \mu t + C_T) - h_*(\cdot)\|_\infty < \epsilon$. Then, we have that

$$\begin{aligned}\|u(t, \cdot + m_t - \mu t) - h_*(\cdot - C)\|_\infty &\le \|u(t, \cdot + m_t - \mu t) - v^T(t, \cdot + m_t - \mu t)\|_\infty \\ &\quad + \|v^T(t, \cdot + m_t - \mu t) - h_*(\cdot - C_T)\|_\infty \\ &\quad + \|h_*(\cdot - C_T) - h_*(\cdot - C)\|_\infty, \\ &\le 2\epsilon + c|C_T - C|,\end{aligned}$$

where $c = \max_{x\in\mathbb{R}} h_*'(x)$. As $C_T \to C$, for T large enough independently of x this can be made smaller than 3ϵ. Thus $\|u(t, \cdot + m_t - \mu t) - h_*(\cdot - C)\|_\infty \to 0$ as $t \to \infty$, which is the Theorem. $\square$

It now remains to prove Lemma 26. We start with

Lemma 27. *For any $\epsilon > 0$ there exists T_ϵ such that for all $t \ge T \ge T_\epsilon$ one has $v^T(t,0) \le \epsilon/(1+\epsilon)$.*

(The $1 + \epsilon$ in the denominator makes the following easier.)

Proof. We use the representation (63). Let $t \ge T$; obviously

$$v^T(t,0) \le \mathbb{P}^0(\min_{u\in\mathcal{N}_{\text{all}}(t)} X_u(t) > 0) = h(t, -\mu t),$$

where h is the solution of (20). $h(t, -\mu t)$ is by definition the probability that the leftmost particle at time t of a driftless branching Brownian motion is to the right of $-\mu t$; it is also the probability that the leftmost particle at time t of a branching Brownian motion with drift μ is to the right of zero. For $\mu < \sqrt{2\beta}$ (regimes A and B), this probability is known to tend to zero when $t \to \infty$. $\square$

The next step is the following Lemma:

Lemma 28. *For any $\epsilon > 0$ and any $T > T_\epsilon$ one has*

$$(1+\epsilon)v^T(t,x) - \epsilon \le u(t,x) \le v^T(t,x), \quad (t,x) \in [T,\infty) \times \mathbb{R}_+.$$

(The T_ϵ in Lemma 28 is the same as in Lemma 27.)

Proof. $u \le v^T$ follows immediately from their definitions as probabilities. Let us introduce

$$\tilde{u}(t,x) := \frac{u(t,x)+\epsilon}{1+\epsilon}.$$

We have that

$$(1+\epsilon)\partial_t \tilde{u} = (1+\epsilon)\frac{1}{2}\partial_{xx}\tilde{u} + (1+\epsilon)\mu\partial_x\tilde{u} + \beta\big[\big((1+\epsilon)\tilde{u}-\epsilon\big)^2 - \big((1+\epsilon)\tilde{u}-\epsilon\big)\big]$$

Performing simple calculations we arrive at

$$\begin{aligned}\partial_t\tilde{u} &= \frac{1}{2}\partial_{xx}\tilde{u} + \mu\partial_x\tilde{u} + \beta(\tilde{u}-1)(\tilde{u}-\epsilon+\epsilon\tilde{u})\\ &\ge \frac{1}{2}\partial_{xx}\tilde{u} + \mu\partial_x\tilde{u} + \beta(\tilde{u}-1)\tilde{u}\end{aligned}$$

since $\tilde{u} \le 1$ and $\epsilon > 0$.

Now, for any $T > T_\epsilon$, we have with Lemma 27

$$v^T(t,0) \le \frac{\epsilon}{1+\epsilon} = \tilde{u}(t,0), \qquad t \ge T.$$

Moreover one checks directly that

$$v^T(T,x) = u(T,x) \le \tilde{u}(T,x), \quad x \ge 0.$$

By the parabolic maximum principle (and the unicity of solutions) [5] we get that for any $T > T_\epsilon$

$$v^T(t,x) \le \tilde{u}(t,x), \quad \forall (t,x) \in (T,\infty)\times\mathbb{R}_+.$$

This proves the first inequality and thus concludes the proof of the lemma. $\square$

Lemma 28 implies that $|u(t,x) - v^T(t,x)| \le \epsilon(1-v^T(t,x)) \le \epsilon$ for each $x \in \mathbb{R}_+$ and each t and T with $t \ge T \ge T(\epsilon)$, which is the first assertion of Lemma 26.

The last step is then to prove that C_T has a limit C for large T.

As $u(t,\cdot)$ is strictly increasing and continuous, $u(t,0)=0$ and $\lim_{x\to+\infty} u(t,x) = 1$, we may define $m_{\frac{1}{2}} : (0,+\infty) \mapsto \mathbb{R}_+$ by

$$u\big(t, m_{\frac{1}{2}}(t)\big) = 1/2.$$

Fix $\epsilon > 0$. We have that

$$\begin{aligned}\big|1/2 - h_*\big(m_{\frac{1}{2}}(t) - m_t + \mu t - C_T\big)\big| &\le \big|u\big(t, m_{\frac{1}{2}}(t)\big) - v^T\big(t, m_{\frac{1}{2}}(t)\big)\big|\\ &\quad + \big|v^T\big(t, m_{\frac{1}{2}}(t)\big) - h_*\big(m_{\frac{1}{2}}(t) - m_t + \mu t - C_T\big)\big|,\\ &\le 2\epsilon,\end{aligned}$$

as long as T and t are large enough by (65) and (66). From this we deduce

$$m_{\frac{1}{2}}(t) - m_t + \mu t - C_T \in \big[h_*^{-1}(1/2-2\epsilon), h_*^{-1}(1/2+2\epsilon)\big]. \tag{67}$$

Consequently

$$\limsup_{t\to+\infty}\big[m_{\frac{1}{2}}(t) - m_t + \mu t\big] - \liminf_{t\to+\infty}\big[m_{\frac{1}{2}}(t) - m_t + \mu t\big] \le h_*^{-1}(1/2+2\epsilon) - h_*^{-1}(1/2-2\epsilon).$$

Since ϵ can be chosen arbitrarily small we have that $\lim_{t\to+\infty}\big[m_{\frac{1}{2}}(t) - m_t + \mu t\big] = C$, for some constant $C \in \mathbb{R}$. This and (67) immediately yields that

$$\lim_{T\to+\infty} C_T = C,$$

where we used that $h_*^{-1}(1/2) = 0$. This concludes the proof of Lemma 26.

3 Radius of convergence and asymptotic behavior of S_0

In Section 1.2.3, we related the $\omega_s(x)$ to a function $x \mapsto \Phi(z)$ defined as a series of which the coefficients a_n follows the recursive equation (11). We write here the same property in a slightly different but equivalent way. Let $p \in (0,1]$ be defined by

$$p := \frac{2\beta}{r^2},$$

and introduce $\Psi^{(p)}(z) = p\Phi(z/p)$ and $b_n^{(p)} = a_n/p^{n-1}$. These quantities satisfy the relations

$$\Psi^{(p)}(z) = \sum_{n\geq 1} b_n^{(p)} z^p, \qquad b_1^{(p)} = 1, \qquad b_n^{(p)} = \frac{1}{(n-1)(n-p)} \sum_{j=1}^{n-1} b_j^{(p)} b_{n-j}^{(p)}, \quad n \geq 2. \tag{68}$$

Let $\mathcal{R}^{(p)}$ be the radius of convergence of $\Psi^{(p)}$. We know that there exists a B_{s_0} relating $\Psi^{(p)}$ and ω_{s_0} through

$$\omega_{s_0}(x) = 1 - \frac{1}{p}\Psi^{(p)}(pB_{s_0}e^{-rx}).$$

The following observation will be useful. Since $\omega'_{s_0}(0) = 0$ and $\omega''_{s_0}(0) < 0$, the function ω_{s_0} (defined on a domain containing zero) has a local maximum in zero. This implies that for $p > 0$ the function $\Psi^{(p)}$ has a local minimum in $m^{(p)} := pB_{s_0} < 0$. In fact $m^{(p)}$ is the first local minimum (and indeed the first point where the first derivative cancels) one encounters left of zero for $\Psi^{(p)}$.

The steps of the proof are the following:

1. We show that $\mathcal{R}^{(p)} \geq 4$ for small enough p (including $p = 0$).
2. We prove that there exists $m^{(0)} \in (-3, 0]$ which is the first minimum one encounters left of zero for $\Psi^{(0)}$ and that

$$[\Psi^{(0)}]'(x) < 0, x \in [a, m^{(0)}), \quad \text{and} \quad [\Psi^{(0)}]'(x) > 0, x \in (m^{(0)}, 0], \tag{69}$$

   for some $a \in [-3, m^{(0)}]$.
3. We show that $[\Psi^{(p)}]'$ converges to $[\Psi^{(p)}]'$ uniformly on $(-3, 0]$. This implies that

$$\lim_{p\searrow 0} m^{(p)} = m^{(0)} \in (-3, 0). \tag{70}$$

4. Since $|pB_{s_0}| \to |m^{(0)}| < 4$ we conclude that that B_{s_0} is within the radius of convergence of Φ for p small enough. The identity

$$s_0 = w_{s_0}(0) = 1 - \Phi(B_{s_0}) = 1 - \Psi^{(p)}(m^{(p)})/p.$$

   shows that

$$\lim_{p\searrow 0} ps_0(p) = \Psi^{(0)}(m^{(0)}),$$

   where we made the dependance of s_0 on $p = 2\beta/r^2$ explicit.

We now prove these points.

1. The key remark is that if for a real $\alpha > 0$ and an integer n_0, one has $b_n^{(p)} \leq (n_0 - p)\alpha^{-n}$ for all $n \in \{1, \ldots, n_0 - 1\}$ then, as can be shown by a very simple recursion, the property $b_n^{(p)} \leq (n_0 - p)\alpha^{-n}$ holds for all $n \geq 1$.

Computing the first values of $b_n^{(0)}$, one checks easily that the maximum of $4^n b_n^{(0)}$ for $n \in \{1, \ldots, 14\}$ is around 14.14. For p small enough, by continuity of $p \mapsto b_n^{(p)}$, the maximum of $b_n^{(p)}$ for $n \in \{1, \ldots, 14\}$ will be no more than $15 - p$ and hence one has

$$b_n^{(p)} \leq 15 \times 4^{-n}, \qquad \text{(for } p \text{ small enough)} \tag{71}$$

As a consequence, $\mathcal{R}^{(p)} \geq 4$ for p small enough (including $p = 0$).

2. The bound (71) applies for $p = 0$. Thus, for any $z \in [-3, 3]$ using only the the fifty first terms of the expansion leads to an error of at most $\sum_{n \geq 51} 15 \times (3/4)^n < 3\,10^{-5}$. In that way we computed $\Psi^{(0)}(-3) \approx -0.8528$ and $\Psi^{(0)}(-2.5) \approx -0.8575$. Therefore $\Psi^{(0)}(-2.5)$ is smaller than both $\Psi^{(0)}(-3)$ and $\Psi^{(0)}(0) = 0$, and the function $\Psi^{(0)}$ must have a minimum in $(-3, 0)$. In other words we proved $m^{(0)} \in (-3, 0)$. It is easy to check that $\sum_{n=1}^{50} n(n-1) a_n x^{n-2} \geq 0.7$ for $x \in [-3, 0]$. Estimating an error by $\sum_{n \geq 51} 15n(n-1)(3/4)^n < 0.074$ we conclude that $[\Psi^{(0)}]''(x) > 0$ for $x \in [-3, 0]$. In this way we get (69).

3. By (71) there exist $p_0 > 0$ and $C > 0$ such that the functions $[\Psi^{(p)}]'$ are analytic in $[-3, 0]$ and $\sup_{p \in [0, p_0], x \in [-3, 0]} |\Psi^{(p)}(x)| < C$. By continuity of $p \mapsto b_n^{(p)}$, for any $x \in [-3, 0]$ we have $[\Psi^{(p)}]'(x) \to [\Psi^{(0)}]'(x)$. The Vitali-Proter theorem strengthen this to uniform convergence. This together with (69) implies easily (70).

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